$$p_n(t_1, \ldots, t_n, z_1, z_2, \ldots) = (ex(t_1), \ldots, ex(t_n), z_1, z_2, \ldots)$$

Let $\tilde{X} = \vee \tilde{X}_n$ and define $p: \tilde{X} \to X$ so that $p \mid \tilde{X}_n = p_n$. The components of $\tilde{X}$ are the spaces $\tilde{X}_n$ and the map $p \mid \tilde{X}_n = p_n: \tilde{X}_n \to X$ is a covering projection. However, p is not a covering projection, because no open subset of X is evenly covered by p.

For later purposes we should like to have the analogues of theorems 11 and 14, in which "component" is replaced by "path component." For this we need the following definition: a topological space is said to be *locally path connected* if the path components of open subsets are open. The following are easy consequences of this definition.

16 *Any open subset of a locally path-connected space is itself locally path connected.* ∎

17 *A locally path-connected space is locally connected.* ∎

18 *In a locally path-connected space the components and path components coincide.* ∎

19 *A connected locally path-connected space is path connected.* ∎

From statements 17 and 18 we obtain the following extension of theorems 11 and 14.

20 THEOREM *If X is locally path connected, a continuous map $p: \tilde{X} \to X$ is a covering projection if and only if for each path component A of X*

$$p \mid p^{-1}A: p^{-1}A \to A$$

is a covering projection. In this case, if $\tilde{A}$ is any path component of $\tilde{X}$, then $p \mid \tilde{A}$ is a covering projection of $\tilde{A}$ onto some path component of X. ∎

2 THE HOMOTOPY LIFTING PROPERTY

The homotopy lifting property is dual to the homotopy extension property. It leads to the concept of fibration, which is dual to that of cofibration introduced in Sec. 1.4. In this section we define the concept of fibration and prove that a covering projection is a special kind of fibration. This special class of fibrations will be regarded as generalized covering projections, and our subsequent study of covering projections will be based on a study of the more general concept. At the end of the chapter we return to the general consideration of fibrations.

We begin with an important problem of algebraic topology, called the lifting problem, which is dual to the extension problem. Let $p: E \to B$ and $f: X \to B$ be maps. The *lifting problem* for f is to determine whether there is a continuous map $f'\colon X \to E$ such that $f = p \circ f'$—that is, whether the dotted arrow in the diagram

$$\begin{array}{ccc} & & E \\ & \nearrow & \downarrow p \\ X & \xrightarrow{f} & B \end{array}$$

corresponds to a continuous map making the diagram commutative. If there is such a map f', then f can be *lifted* to E, and we call f' a *lifting*, or *lift*, of f. In order that the lifting problem be a problem in the homotopy category, we need an analogue of the homotopy extension property, called the homotopy lifting property, defined as follows. A map $p\colon E \to B$ is said to have the *homotopy lifting property with respect to a space* X if, given maps $f'\colon X \to E$ and $F\colon X \times I \to B$ such that $F(x,0) = pf'(x)$ for $x \in X$, there is a map $F'\colon X \times I \to E$ such that $F'(x,0) = f'(x)$ for $x \in X$ and $p \circ F' = F$. If f' is regarded as a map of $X \times 0$ to E, the existence of F' is equivalent to the existence of a map represented by the dotted arrow that makes the following diagram commutative:

$$\begin{array}{ccc} X \times 0 & \xrightarrow{f'} & E \\ \downarrow \cap & \nearrow & \downarrow p \\ X \times I & \xrightarrow{F} & B \end{array}$$

If $p\colon E \to B$ has the homotopy lifting property with respect to X and $f_0, f_1\colon X \to B$ are homotopic, it is easy to see that f_0 can be lifted to E if and only if f_1 can be lifted to E. Hence, whether or not a map $X \to B$ can be lifted to E is a property of the homotopy class of the map. Thus the homotopy lifting property implies that the lifting problem for maps $X \to B$ is a problem in the homotopy category.

A map $p\colon E \to B$ is called a *fibration* (or *Hurewicz fiber space* in the literature) if p has the homotopy lifting property with respect to every space. E is called the *total space* and B the *base space* of the fibration. For $b \in B$, $p^{-1}(b)$ is called the *fiber over* b.

If $p\colon E \to B$ is a fibration, any path ω in B such that $\omega(0) \in p(E)$ can be lifted to a path in E. In fact, ω can be regarded as a homotopy $\omega\colon P \times I \to B$ where P is a one-point space, and a point $e_0 \in E$ such that $p(e_0) = \omega(0)$ corresponds to a map $f\colon P \to E$ such that $pf(P) = \omega(P,0)$. It follows from the homotopy lifting property of p that there exists a path $\tilde\omega$ in E such that $\tilde\omega(0) = e_0$ and $p \circ \tilde\omega = \omega$. Then $\tilde\omega$ is a lifting of ω.

1 EXAMPLE Let F be any space and let $p\colon B \times F \to B$ be the projection to the first factor. Then p is a fibration, and for any $b \in B$ the fiber over b is homeomorphic to F.

To prove that a covering projection is a fibration, we first establish the following *unique-lifting property* of covering projections for connected spaces.

2 THEOREM *Let $p\colon \tilde{X} \to X$ be a covering projection and let $f, g\colon Y \to \tilde{X}$ be liftings of the same map (that is, $p \circ f = p \circ g$). If Y is connected and f agrees with g for some point of Y, then $f = g$.*

PROOF Let $Y_1 = \{y \in Y \mid f(y) = g(y)\}$. We show that Y_1 is open in Y. If $y \in Y_1$, let U be an open neighborhood of $pf(y)$ evenly covered by p and let $\tilde{U}$ be an open subset of $\tilde{X}$ containing $f(y)$ such that p maps $\tilde{U}$ homeomorphically onto U. Then $f^{-1}(\tilde{U}) \cap g^{-1}(\tilde{U})$ is an open subset of Y containing y and contained in Y_1.

Let $Y_2 = \{y \in Y \mid f(y) \neq g(y)\}$. We show that Y_2 is also open in Y (if $\tilde{X}$ were assumed to be Hausdorff, this would follow from a general property of Hausdorff spaces). Let $y \in Y_2$ and let U be an open neighborhood of $pf(y)$ evenly covered by p. Since $f(y) \neq g(y)$, there are disjoint open subsets $\tilde{U}_1$ and $\tilde{U}_2$ of $\tilde{X}$ such that $f(y) \in \tilde{U}_1$ and $g(y) \in \tilde{U}_2$ and p maps each of the sets $\tilde{U}_1$ and $\tilde{U}_2$ homeomorphically onto U. Then $f^{-1}(\tilde{U}_1) \cap g^{-1}(\tilde{U}_2)$ is an open subset of Y containing y and contained in Y_2.

Since $Y = Y_1 \cup Y_2$ and Y_1 and Y_2 are disjoint open sets, it follows from the connectedness of Y that either $Y_1 = \varnothing$ or $Y_1 = Y$. By hypothesis, $Y_1 \neq \varnothing$, so $Y = Y_1$ and $f = g$. ∎

We are now ready to prove that a covering projection has the homotopy lifting property.

3 THEOREM *A covering projection is a fibration.*

PROOF Let $p\colon \tilde{X} \to X$ be a covering projection and let $f'\colon Y \to \tilde{X}$ and $F\colon Y \times I \to X$ be maps such that $F(y,0) = pf'(y)$ for $y \in Y$. We show that for each $y \in Y$ there is an open neighborhood N_y of y in Y and a map $F'_y\colon N_y \times I \to \tilde{X}$ such that $F'_y(y',0) = f'(y')$ for $y' \in N_y$ and $pF'_y = F \mid N_y \times I$. Assume that we have such neighborhoods N_y and maps F'_y. If $y'' \in N_y \cap N_{y'}$, then $F'_y \mid y'' \times I$ and $F'_{y'} \mid y'' \times I$ are maps of the connected space $y'' \times I$ into $\tilde{X}$ such that for $t \in I$

$$p \circ (F'_y \mid y'' \times I)(y'',t) = F(y'',t) = p \circ (F'_{y'} \mid y'' \times I)(y'',t)$$

Because $(F'_y \mid y'' \times I)(y'',0) = f'(y'') = (F'_{y'} \mid y'' \times I)(y'',0)$, it follows from theorem 2 that $F'_y \mid y'' \times I = F'_{y'} \mid y'' \times I$. Since this is true for all $y'' \in N_y \cap N_{y'}$, it follows that $F'_y \mid (N_y \cap N_{y'}) \times I = F'_{y'} \mid (N_y \cap N_{y'}) \times I$. Hence there is a continuous map $F'\colon Y \times I \to \tilde{X}$ such that $F' \mid N_y \times I = F'_y$, and F' is a lifting of F such that $F'(y,0) = f'(y)$ for $y \in Y$. Thus we have reduced the theorem to the construction of the open neighborhoods N_y and maps F'_y.

It follows from the fact that $p\colon \tilde{X} \to X$ is a covering projection (and the compactness of I) that for each $y \in Y$ there is an open neighborhood N_y of y and a sequence $0 = t_0 < t_1 < \cdots < t_m = 1$ of points of I such that for $i = 1, \ldots, m$, $F(N_y \times [t_{i-1},t_i])$ is contained in some open subset of X evenly covered by p. We show that there is a map $F'_y\colon N_y \times I \to \tilde{X}$ with the desired properties. It suffices to define maps

$$G_i\colon N_y \times [t_{i-1},t_i] \to \tilde{X} \qquad i = 1, \ldots, m$$

such that

$$\begin{aligned} p \circ G_i &= F \mid N_y \times [t_{i-1},t_i] \\ G_1(y',0) &= f'(y') && y' \in N_y \\ G_{i-1}(y',t_{i-1}) &= G_i(y',t_{i-1}) && y' \in N_y,\ i = 2, \ldots, m \end{aligned}$$

because, given such maps G_i, there is a map $F'_y\colon N_y \times I \to \tilde{X}$ such that $F'_y \mid N_y \times [t_{i-1},t_i] = G_i$ for $i = 1, \ldots, m$. Then F'_y has the desired properties.

The maps G_i are defined by induction on i. To define G_1, let U be an open subset of X evenly covered by p such that $F(N_y \times [t_0,t_1]) \subset U$. Let $\{\tilde{U}_j\}$ be a collection of disjoint open subsets of $\tilde{X}$ such that $p^{-1}(U) = \cup\, \tilde{U}_j$ and p maps $\tilde{U}_j$ homeomorphically onto U for each j. Let $V_j = f'^{-1}(\tilde{U}_j)$. Then $\{V_j\}$ is a collection of disjoint open sets covering N_y, and G_1 is defined to be the unique map such that for each j, G_1 maps $V_j \times [t_0,t_1]$ into $\tilde{U}_j$ to be a lifting of $F \mid V_j \times [t_0,t_1]$. This defines G_1.

Assume G_{i-1} defined for $1 < i \leq m$. Let U' be an open subset of X evenly covered by p such that $F(N_y \times [t_{i-1},t_i]) \subset U'$. Let $\{\tilde{U}'_k\}$ be a collection of disjoint open subsets of $\tilde{X}$ such that $p^{-1}(U') = \cup\, \tilde{U}'_k$ and p maps $\tilde{U}'_k$ homeomorphically onto U' for each k. Let $V'_k = \{y' \in N_y \mid G_{i-1}(y',t_{i-1}) \in \tilde{U}'_k\}$. Then $\{V'_k\}$ is a collection of disjoint open sets covering N_y, and G_i is defined to be the unique map such that for each k, G_i maps $V'_k \times [t_{i-1},t_i]$ into $\tilde{U}'_k$ to be a lifting of $F \mid V'_k \times [t_{i-1},t_i]$. This defines G_i. ∎

A map $p\colon E \to B$ is said to have *unique path lifting* if, given paths ω and ω' in E such that $p \circ \omega = p \circ \omega'$ and $\omega(0) = \omega'(0)$, then $\omega = \omega'$. It follows from theorem 2 that a covering projection has unique path lifting.

4 LEMMA *If a map has unique path lifting, it has the unique-lifting property for path-connected spaces.*

PROOF Assume that $p\colon E \to B$ has unique path lifting. Let Y be path connected and suppose that $f, g\colon Y \to E$ are maps such that $p \circ f = p \circ g$ and $f(y_0) = g(y_0)$ for some $y_0 \in Y$. We must show $f = g$. Let $y \in Y$ and let ω be a path in Y from y_0 to y. Then $f \circ \omega$ and $g \circ \omega$ are paths in E that are liftings of the same path in B and have the same origin. Because p has unique path lifting, $f \circ \omega = g \circ \omega$. Therefore

$$f(y) = (f \circ \omega)(1) = (g \circ \omega)(1) = g(y) \quad ∎$$

The following theorem characterizes fibrations with unique path lifting.

5 THEOREM *A fibration has unique path lifting if and only if every fiber has no nonconstant paths.*

PROOF Assume that $p\colon E \to B$ is a fibration with unique path lifting. Let ω be a path in the fiber $p^{-1}(b)$ and let ω' be the constant path in $p^{-1}(b)$ such that $\omega'(0) = \omega(0)$. Then $p \circ \omega = p \circ \omega'$, which implies $\omega = \omega'$. Hence ω is a constant path.

Conversely, assume that $p: E \to B$ is a fibration such that every fiber has no nontrivial path and let ω and ω' be paths in E such that $p \circ \omega = p \circ \omega'$ and $\omega(0) = \omega'(0)$. For $t \in I$, let ω_t'' be the path in E defined by

$$\omega_t''(t') = \begin{cases} \omega((1 - 2t')t) & 0 \le t' \le \frac{1}{2} \\ \omega'((2t' - 1)t) & \frac{1}{2} \le t' \le 1 \end{cases}$$

Then ω_t'' is a path in E from $\omega(t)$ to $\omega'(t)$, and $p \circ \omega_t''$ is a closed path in B that is homotopic relative to $\dot{I}$ to the constant path at $p\omega(t)$. By the homotopy lifting property of p, there is a map $F': I \times I \to E$ such that $F'(t',0) = \omega_t''(t')$ and F' maps $0 \times I \cup I \times 1 \cup 1 \times I$ to the fiber $p^{-1}(p\omega(t))$. Because $p^{-1}(p\omega(t))$ has no nonconstant paths, F' maps $0 \times I$, $I \times 1$, and $1 \times I$ to a single point. It follows that $F'(0,0) = F'(1,0)$. Therefore $\omega_t''(0) = \omega_t''(1)$ and $\omega(t) = \omega'(t)$. ■

We have seen that a covering projection is a fibration with unique path lifting. It will be shown in Sec. 2.4 that if the base space satisfies some mild hypotheses, any fibration with unique path lifting is a covering projection. One reason for studying fibrations with unique path lifting as generalized covering projections is that the following two theorems are easily proved, but both are false for covering projections.

6 THEOREM *The composite of fibrations (with unique path lifting) is a fibration (with unique path lifting).* ■

7 THEOREM *The product of fibrations (with unique path lifting) is a fibration (with unique path lifting).* ■

An example shows that theorem 6 is false for covering projections.

8 EXAMPLE Let X and X_n, for $n \ge 1$, be a countable product of 1-spheres. Let $\tilde{X}_n = \mathbf{R}^n \times X$ and define $p_n: \tilde{X}_n \to X_n$ by

$$p_n(t_1, \ldots, t_n, z_1, z_2, \ldots) = (ex(t_1), \ldots, ex(t_n), z_1, z_2, \ldots)$$

Then p_n is a covering projection for $n \ge 1$. It follows from theorem 2.1.11 that $\vee p_n: \vee \tilde{X}_n \to \vee X_n$ is a covering projection. Since $\vee X_n$ is the product of X and the set of positive integers, there is a covering projection $\vee X_n \to X$ (see example 2.1.2). The composite

$$\vee \tilde{X}_n \to \vee X_n \to X$$

is not a covering projection (*cf.* example 2.1.15).

Similarly, theorem 7 is false for covering projections.

9 EXAMPLE For $n \ge 1$, let $p_n: \tilde{X}_n \to X_n$ be the covering projection $ex: \mathbf{R} \to S^1$. Then

$$\times p_n: \times \tilde{X}_n \to \times X_n$$

is not a covering projection.

It follows from theorem 6 that there is a category whose objects are topological spaces and whose morphisms are fibrations with unique path lifting. We shall now describe a category, depending on a given base space, which is of more use in studying covering projections or fibrations. For a given space X there is a category whose objects are maps $p: \tilde{X} \to X$, which are fibrations with unique path lifting, and whose morphisms are commutative triangles

$$\begin{array}{ccc} X_1 & \xrightarrow{f} & X_2 \\ & {\scriptstyle p_1}\searrow \quad \swarrow{\scriptstyle p_2} & \\ & X & \end{array}$$

If $p_j: \tilde{X}_j \to X$ is an indexed family of objects in this category, let $p: \vee \tilde{X}_j \to X$ be the map such that $p \mid \tilde{X}_j = p_j$. Then p is also an object in the category and is the sum of the collection $\{p_j\}$ in the category.

To show that this category also has products, given maps $p_j: \tilde{X}_j \to X$, let

$$\tilde{X} = \{(x_j) \in \times \tilde{X}_j \mid p_j(x_j) = p_{j'}(x_{j'}) \text{ for all } j, j'\}$$

and define $p: \tilde{X} \to X$ by $p((x_j)) = p_j(x_j)$. If each p_j is a fibration, so is p, and if each p_j has unique path lifting, so does p. Hence p is a product of $\{p_j\}$ in the category of fibrations with unique path lifting. This map p is called the *fibered product* of the maps $\{p_j\}$. We consider it in more detail in Sec. 2.8.

There is a similar category whose objects are covering projections with base space X and whose morphisms are commutative triangles. This category has finite sums and finite products, but neither arbitrary sums nor arbitrary products. In fact, for each n let

$$p_n: \mathbf{R}^n \times S^1 \times S^1 \times \cdots \to S^1 \times S^1 \times \cdots$$

be defined by $p_n(t_1, \ldots, t_n, z_1, z_2, \ldots) = (e^{2\pi i t_1}, \ldots, e^{2\pi i t_n}, z_1, z_2, \ldots)$, as in example 8. Then the collection $\{p_n\}$ has neither a sum nor a product in the category of covering projections with base space X.

3 RELATIONS WITH THE FUNDAMENTAL GROUP

In a fibration with unique path lifting the fundamental group of the total space is isomorphic to a subgroup of the fundamental group of the base space. The corresponding subgroup of the fundamental group will lead to a classification of fibrations with unique path lifting. In fact, we shall see in the next section that the fundamental group functor solves the lifting problem for fibrations with unique path lifting. The present section is devoted to consideration of the relation between the fundamental groups of the total space and the base space of a fibration with unique path lifting.

We begin with a localization property for fibrations which is an analogue of theorem 2.1.14.

1 LEMMA *Let $p: E \to B$ be a fibration. If A is any path component of E, then pA is a path component of B and $p \mid A: A \to pA$ is a fibration.*

PROOF Since pA is the continuous image of a path-connected space, it is path connected. It is a maximal path-connected subset of B, for if ω is a path in B that begins in pA, there is a lifting $\tilde\omega$ of ω that begins in A. Since A is a path component of E, $\tilde\omega$ is a path in A. Therefore $\omega = p \circ \tilde\omega$ is a path in pA. Hence pA is a maximal path-connected subset of B and, by theorem 1.7.9, a path component of B.

To show that $p \mid A: A \to pA$ has the homotopy lifting property, let $f': Y \to A$ and $F: Y \times I \to pA$ be maps such that $F(y,0) = pf'(y)$. Because p is a fibration, there is a map $F': Y \times I \to E$ such that $p \circ F' = F$ and $F'(y,0) = f'(y)$. For any $y \in Y$, F' must map $y \times I$ into the path component of E containing $F'(y,0)$. Therefore $F'(y \times I) \subset A$ for all y, and $F': Y \times I \to A$ is a lifting of F such that $F'(y,0) = f'(y)$. ∎

For locally path-connected spaces we have the following analogue of theorem 2.1.20, which reduces the study of fibrations to the study of fibrations with total space and base space path connected.

2 THEOREM *Let $p: E \to B$ be a map. If E is locally path connected, p is a fibration if and only if for each path component A of E, pA is a path component of B and $p \mid A: A \to pA$ is a fibration.*

PROOF If $p: E \to B$ is a fibration and A is a path component of E, it follows from lemma 1 that pA is a path component of B and $p \mid A: A \to pA$ is a fibration.

To prove the converse, let $f': Y \to E$ and $F: Y \times I \to B$ be such that $F(y,0) = f'(y)$. Let $\{A_j\}$ be the path components of E. Then $\{A_j\}$ are disjoint open subsets of E. Let $V_j = f'^{-1}(A_j)$. The collection $\{V_j\}$ is a disjoint open covering of Y. Therefore, to construct a map $F': Y \times I \to E$ such that $p \circ F' = F$ and $F'(y,0) = f'(y)$, it suffices to construct maps $F'_j: V_j \times I \to E$ for all j such that $p \circ F'_j = F \mid V_j \times I$ and $F_j(y,0) = f'(y,0)$.

Because $F(y \times I)$ is contained in the path component of B containing $F(y,0) = pf'(y)$, it follows from the fact that pA_j is a path component of B that $F(V_j \times I) \subset pA_j$ for all j. Because $p \mid A_j: A_j \to pA_j$ is a fibration, there is a map $F'_j: V_j \times I \to A_j$ such that $pF'_j = F \mid V_j \times I$ and $F'_j(y,0) = f'(y)$ for $y \in V_j$. Therefore p has the homotopy lifting property. ∎

Since every path in a topological space lies in some path component of the space, it is clear that theorem 2 remains valid if the term "fibration" is replaced throughout by "fibration with unique path lifting."

The main result on fibrations with unique path lifting is embodied in the following statement.

3 LEMMA *Let $p: \tilde X \to X$ be a fibration with unique path lifting. If ω and ω' are paths in $\tilde X$ such that* $\omega(0) = \omega'(0)$ and $p \circ \omega \simeq p \circ \omega'$, *then $\omega \simeq \omega'$.*

PROOF Let $F: I \times I \to X$ be a homotopy relative to $\dot{I}$ from $p \circ \omega$ to $p \circ \omega'$ [that is, $F(t,0) = p\omega(t)$ and $F(t,1) = p\omega'(t)$, and $F(0,t) = p\omega(0)$ and $F(1,t) = p\omega(1)$]. By the homotopy lifting property of fibrations, there is a map $F': I \times I \to \tilde{X}$ such that $F'(t,0) = \omega(t)$ and $p \circ F' = F$. Then $F'(0 \times I)$ and $F'(1 \times I)$ are contained in $p^{-1}(p\omega(0))$ and $p^{-1}(p\omega(1))$, respectively. By theorem 2.2.5, $F'(0 \times I)$ and $F'(1 \times I)$ are single points. Hence F' is a homotopy relative to $\dot{I}$ from ω to some path ω'' such that $\omega''(0) = \omega(0)$ and $p \circ \omega'' = p \circ \omega'$. Since $\omega'(0) = \omega(0)$, it follows from the unique-path-lifting property of p that $\omega' = \omega''$ and $F': \omega \simeq \omega'$ rel $\dot{I}$. ■

It follows from lemma 3 that if $p: \tilde{X} \to X$ is a fibration with unique path lifting, then for any two objects $\tilde{x}_0$ and $\tilde{x}_1$ in the fundamental groupoid of $\tilde{X}$, $p_\#$ maps hom $(\tilde{x}_0,\tilde{x}_1)$ injectively into hom $(p(\tilde{x}_0),p(\tilde{x}_1))$. In particular, if $\tilde{x}_0 = \tilde{x}_1$, we obtain the following theorem.

4 THEOREM *Let $p: \tilde{X} \to X$ be a fibration with unique path lifting. For any $\tilde{x}_0 \in \tilde{X}$ the homomorphism.*

$$p_\#: \pi(\tilde{X},\tilde{x}_0) \to \pi(X,x_0)$$

is a monomorphism. ■

This last result provides the basis for the reduction of problems concerning fibrations with unique path lifting to problems about the fundamental group. In order that the fundamental group be really representative of the space in question, we assume that the spaces involved are path connected. It follows from theorem 2 that this is no loss of generality for locally path-connected spaces.

5 LEMMA *Let $p: \tilde{X} \to X$ be a fibration with unique path lifting and assume that $\tilde{X}$ is a nonempty path-connected space. If $\tilde{x}_0$, $\tilde{x}_1 \in \tilde{X}$, there is a path ω in X from $p(\tilde{x}_0)$ to $p(\tilde{x}_1)$ such that*

$$p_\#\pi(\tilde{X},\tilde{x}_0) = h_{[\omega]}p_\#\pi(\tilde{X},\tilde{x}_1)$$

Conversely, given a path ω in X from $p(\tilde{x}_0)$ to x_1, there is a point $\tilde{x}_1 \in p^{-1}(x_1)$ such that

$$h_{[\omega]}p_\#\pi(\tilde{X},\tilde{x}_1) = p_\#\pi(\tilde{X},\tilde{x}_0)$$

PROOF For the first part, let $\tilde{\omega}$ be a path in $\tilde{X}$ from $\tilde{x}_0$ to $\tilde{x}_1$. Then $\pi(\tilde{X},\tilde{x}_0) = h_{[\tilde{\omega}]}\pi(\tilde{X},\tilde{x}_1)$. Therefore

$$p_\#\pi(\tilde{X},\tilde{x}_0) = h_{[p\circ\tilde{\omega}]}p_\#\pi(\tilde{X},\tilde{x}_1)$$

and so $p \circ \tilde{\omega}$ will do as the path from $p(\tilde{x}_0)$ to $p(\tilde{x}_1)$.

Conversely, given a path ω in X from $p(\tilde{x}_0)$ to x_1, let $\tilde{\omega}$ be a path in $\tilde{X}$ such that $\tilde{\omega}(0) = \tilde{x}_0$ and $p\tilde{\omega} = \omega$. If $\tilde{x}_1 = \tilde{\omega}(1)$, then

$$h_{[\omega]}p_\#\pi(\tilde{X},\tilde{x}_1) = p_\#(h_{[\tilde{\omega}]}\pi(\tilde{X},\tilde{x}_1)) = p_\#\pi(\tilde{X},\tilde{x}_0) \quad ■$$

This easily implies the following result.

6 THEOREM *Let p: $\tilde{X} \to X$ be a fibration with unique path lifting and assume that $\tilde{X}$ is a nonempty path-connected space. For $x_0 \in p\tilde{X}$ the collection $\{p_\#\pi(\tilde{X},\tilde{x}_0) \mid \tilde{x}_0 \in p^{-1}(x_0)\}$ is a conjugacy class in $\pi(X,x_0)$. If ω is a path in $p\tilde{X}$ from x_0 to x_1, then $h_{[\omega]}$ maps the conjugacy class in $\pi(X,x_1)$ to the conjugacy class in $\pi(X,x_0)$.* ∎

Let p: $\tilde{X} \to X$ be a fibration and let ω be a path in X beginning at x_0. Define a map F_ω: $p^{-1}(x_0) \times I \to X$ by $F_\omega(\tilde{x},t) = \omega(t)$ and let i: $p^{-1}(x_0) \subset \tilde{X}$. Then $pi(\tilde{x}) = F_\omega(\tilde{x},0)$ for $\tilde{x} \in p^{-1}(\tilde{x}_0)$. It follows from the homotopy lifting property of p that there exists a map G_ω: $p^{-1}(x_0) \times I \to \tilde{X}$ such that $G_\omega(\tilde{x},0) = i(\tilde{x}) = \tilde{x}$ and $p \circ G_\omega = F_\omega$.

Suppose now that p has unique path lifting. We prove that the map $\tilde{x} \to G_\omega(\tilde{x},1)$ of $p^{-1}(x_0)$ to $p^{-1}(\omega(1))$ depends only on the path class of ω. If $\omega' \simeq \omega$ and $G'_{\omega'}$: $p^{-1}(x_0) \times I \to \tilde{X}$ is a map such that $G'_{\omega'}(\tilde{x},0) = \tilde{x}$ and $p \circ G'_{\omega'} = F_{\omega'}$, then for any $\tilde{x} \in p^{-1}(x_0)$, let $\tilde{\omega}$ and $\tilde{\omega}'$ be the paths in $\tilde{X}$ defined by $\tilde{\omega}(t) = G_\omega(\tilde{x},t)$ and $\tilde{\omega}'(t) = G'_{\omega'}(\tilde{x},t)$. Then $\tilde{\omega}$ and $\tilde{\omega}'$ begin at $\tilde{x}$ and

$$p \circ \tilde{\omega} = \omega \simeq \omega' = p \circ \tilde{\omega}'$$

It follows from lemma 3 that $\tilde{\omega} \simeq \tilde{\omega}'$. Then $G_\omega(\tilde{x},1) = G'_{\omega'}(\tilde{x},1)$ for every $\tilde{x} \in p^{-1}(x_0)$. Therefore there is a well-defined continuous map

$$f_{[\omega]}: p^{-1}(\omega(0)) \to p^{-1}(\omega(1))$$

defined by $f_{[\omega]}(\tilde{x}) = G_\omega(\tilde{x},1)$, where G_ω is as above. It is clear that if $\omega(1) = \omega'(0)$, then $f_{[\omega]*[\omega']} = f_{[\omega']} \circ f_{[\omega]}$.

7 THEOREM *Let p: $\tilde{X} \to X$ be a fibration with unique path lifting. There is a contravariant functor from the fundamental groupoid of X to the category of topological spaces and maps which assigns to $x \in X$ the fiber over x and to $[\omega]$ the function $f_{[\omega]}$.* ∎

The fact that $f_{[\omega]}$ is a homeomorphism for every $[\omega]$ leads to the following corollary.

8 COROLLARY *If p: $\tilde{X} \to X$ is a fibration with unique path lifting and X is path connected, then any two fibers are homeomorphic.* ∎

If X is path connected and p: $\tilde{X} \to X$ is a fibration with unique path lifting, the *number of sheets* of p (or the *multiplicity* of p) is defined to be the cardinal number of $p^{-1}(x)$ (which is independent of $x \in X$, by corollary 8). For a path-connected total space, the multiplicity is determined by the conjugacy class as follows.

9 THEOREM *Let p: $\tilde{X} \to X$ be a fibration with unique path lifting and assume $\tilde{X}$ and X to be nonempty path-connected spaces. If $\tilde{x}_0 \in \tilde{X}$, the multiplicity of p is the index of $p_\#\pi(\tilde{X},\tilde{x}_0)$ in $\pi(X,p(\tilde{x}_0))$.*

PROOF By theorem 7, $\pi(X,p(\tilde{x}_0))$ acts as a group of transformations on the right on $p^{-1}(p(\tilde{x}_0))$ by $\tilde{x} \circ [\omega] = f_{[\omega]}(\tilde{x})$ for $\tilde{x} \in p^{-1}(p(\tilde{x}_0))$. If $\tilde{x}_1, \tilde{x}_2 \in p^{-1}(p(\tilde{x}_0))$, let $\tilde{\omega}$ be a path in $\tilde{X}$ from $\tilde{x}_1$ to $\tilde{x}_2$. Then $[p \circ \tilde{\omega}] \in \pi(X,p(\tilde{x}_0))$ and $\tilde{x}_1 \circ [p\tilde{\omega}] = \tilde{x}_2$.

Therefore $\pi(X,p(\tilde{x}_0))$ acts transitively on $p^{-1}(p(\tilde{x}_0))$. The isotropy group of $\tilde{x}_0$ [that is, the subgroup of $\pi(X,p(\tilde{x}_0))$ leaving $\tilde{x}_0$ fixed] is clearly equal to $p_\#\pi(\tilde{X},\tilde{x}_0)$. From general considerations[1] there is a bijection between the set of right cosets of $p_\#\pi(\tilde{X},\tilde{x}_0)$ in $\pi(X,p(\tilde{x}_0))$ and $p^{-1}(p(\tilde{x}_0))$. ■

10 EXAMPLE For $n \geq 2$ the covering $p\colon S^n \to P^n$ of example 2.1.5 has multiplicity 2. Because S^n is simply connected, $\pi(P^n) \approx \mathbf{Z}_2$ for $n \geq 2$.

A fibration $p\colon \tilde{X} \to X$ with unique path lifting is said to be *regular* if, given any closed path ω in X, either every lifting of ω is closed or none is closed.

11 THEOREM *Let $p\colon \tilde{X} \to X$ be a fibration with unique path lifting. p is regular if and only if $p_\#\pi(\tilde{X},\tilde{x}_0) = p_\#\pi(\tilde{X},\tilde{x}_1)$ whenever $p(\tilde{x}_0) = p(\tilde{x}_1)$.*

PROOF Assume that p is regular and let $\tilde{\omega}$ be a closed path in $\tilde{X}$ at $\tilde{x}_0$. Then $\tilde{\omega}$ is a closed lifting of $p\tilde{\omega}$. Therefore there is a closed lifting $\tilde{\omega}_1$ of $p\tilde{\omega}$ at $\tilde{x}_1$. It follows that $p_\#[\tilde{\omega}] = [p\tilde{\omega}] = p_\#[\tilde{\omega}_1]$. Therefore $p_\#\pi(\tilde{X},\tilde{x}_0) \subset p_\#\pi(\tilde{X},\tilde{x}_1)$. Since the roles of $\tilde{x}_0$ and $\tilde{x}_1$ can be interchanged, it follows that $p_\#\pi(\tilde{X},\tilde{x}_0) = p_\#\pi(\tilde{X},\tilde{x}_1)$.

Conversely, if $p_\#\pi(\tilde{X},\tilde{x}_0) = p_\#\pi(\tilde{X},\tilde{x}_1)$ whenever $p(\tilde{x}_0) = p(\tilde{x}_1)$, let ω be a closed path in X at $p(\tilde{x}_0)$ having a closed lifting $\tilde{\omega}$ at $\tilde{x}_0$. Then

$$[\omega] = p_\#[\tilde{\omega}] \in p_\#\pi(\tilde{X},\tilde{x}_0) = p_\#\pi(\tilde{X},\tilde{x}_1)$$

Therefore there is a closed path $\tilde{\omega}_1$ in $\tilde{X}$ at $\tilde{x}_1$ such that $p\tilde{\omega}_1 \simeq \omega$. If $\tilde{\omega}_1'$ is a lifting of ω such that $\tilde{\omega}_1'(0) = \tilde{x}_1$, then by the unique-path-lifting property of p, $\tilde{\omega}_1 = \tilde{\omega}_1'$. Therefore $\tilde{\omega}_1'$ is a closed lifting of ω at $\tilde{x}_1$ and p is regular. ■

In case $\tilde{X}$ is a nonempty path-connected space, theorems 6 and 11 give the following result.

12 THEOREM *Let $p\colon \tilde{X} \to X$ be a fibration with unique path lifting and assume that $\tilde{X}$ is a nonempty path-connected space. Then p is regular if and only if for some $\tilde{x}_0 \in \tilde{X}_0$, $p_\#\pi(\tilde{X},\tilde{x}_0)$ is a normal subgroup of $\pi(X,p(\tilde{x}_0))$.* ■

4 THE LIFTING PROBLEM

In this section we show that the fundamental group functor solves the lifting problem for fibrations with unique path lifting. As a consequence of this, the fundamental group functor provides a classification of covering projections, which is discussed in the next section.

Our first result is that any map of a contractible space to the base space of a fibration can be lifted.

1 LEMMA *Let $p\colon E \to B$ be a fibration. Any map of a contractible space to B whose image is contained in $p(E)$ can be lifted to E.*

[1] Whenever a group G acts transitively on the right on a set S there is induced a bijection between the set of right cosets of the isotropy group (of any $s \in S$) in G and the set S.

PROOF Let Y be contractible and let $f\colon Y \to B$ be a map such that $f(Y) \subset p(E)$. Because Y is contractible, f is homotopic to a constant map of Y to some point of $f(Y)$. $f(Y) \subset p(E)$, so this constant map can be lifted to E. The homotopy lifting property then implies that f can be lifted to E. ∎

Because we use the fundamental group functor, it will prove technically simpler to consider the lifting problem for spaces with base points.

2 LEMMA *Let $p\colon (\tilde{X},\tilde{x}_0) \to (X,x_0)$ be a fibration with unique path lifting. If y_0 is a strong deformation retract of Y, any map $(Y,y_0) \to (X,x_0)$ can be lifted to a map $(Y,y_0) \to (\tilde{X},\tilde{x}_0)$.*

PROOF Let $f\colon (Y,y_0) \to (X,x_0)$ be a map. f is homotopic relative to y_0 to the constant map $Y \to x_0$. The constant map can be lifted to the constant map $Y \to \tilde{x}_0$. By the homotopy lifting property, f can be lifted to a map $f'\colon Y \to \tilde{X}$ such that f' is homotopic to the constant map $Y \to \tilde{x}_0$ by a homotopy which maps $y_0 \times I$ to $p^{-1}(x_0)$. Because $p^{-1}(x_0)$ has no nonconstant path by theorem 2.2.5, $f'(y_0) = \tilde{x}_0$. ∎

We shall apply lemma 2 to a contractible space in order to lift certain quotient spaces of the contractible space. The usual way to represent a space as the quotient space of a contractible space is to show it is a quotient space of its path space. Given $y_0 \in Y$, the *path space* $P(Y,y_0)$ is the space of continuous maps $\omega\colon (I,0) \to (Y,y_0)$ topologized by the compact-open topology. There is a function $\varphi\colon P(Y,y_0) \to Y$ defined by $\varphi(\omega) = \omega(1)$. If U is an open set in Y,

$$\varphi^{-1}(U) = \langle 1;U\rangle = \{\omega \in P(Y,y_0) \mid \omega(1) \in U\}$$

is an open set in $P(Y,y_0)$. Therefore φ is continuous.

3 LEMMA *The constant path at y_0 is a strong deformation retract of the path space $P(Y,y_0)$.*

PROOF A strong deformation retraction $F\colon P(Y,y_0) \times I \to P(Y,y_0)$ to the constant path at y_0 is defined by

$$F(\omega,t)(t') = \omega((1-t)t') \qquad \omega \in P(Y,y_0);\ t,\ t' \in I \quad ∎$$

We have shown that φ is a continuous map of the contractible path space $P(Y,y_0)$ to Y. If Y is path connected, φ is clearly surjective. If Y is also locally path connected, the following theorem shows that φ is a quotient projection.

4 THEOREM *A connected locally path-connected space Y is the quotient space of its path space $P(Y,y_0)$ by the map φ.*

PROOF We know that φ is continuous, and because a connected locally path-connected space is path connected, it is surjective. To complete the proof it suffices to show that φ is an open map. Let $\omega \in P(Y,y_0)$ and let $W = \bigcap_{1\le i\le n}\langle K_i;U_i\rangle$ be a neighborhood of ω, where K_i is compact in I and U_i is open in Y. We enumerate the K's so that for some $0 \le k \le n$, $1 \in K_1 \cap \cdots \cap K_k$ and

$1 \notin K_{k+1} \cup \cdots \cup K_n$. Because $\omega(1) \in U_1 \cap \cdots \cap U_k$, there is a path-connected neighborhood V of $\omega(1)$ contained in $U_1 \cap \cdots \cap U_k$. Choose $0 < t' < 1$ such that $[t',1] \cap (K_{k+1} \cup \cdots \cup K_n) = \varnothing$ and $\omega([t',1]) \subset V$.

To prove that $\varphi(W) \supset V$, which completes the proof, let $y' \in V$ and let ω' be a path in V from $\omega(t')$ to y'. Define $\bar\omega$: $I \to Y$ by

$$\bar\omega(t) = \begin{cases} \omega(t) & 0 \le t \le t' \\ \omega'\left(\dfrac{t-t'}{1-t'}\right) & t' \le t \le 1 \end{cases}$$

For $i > k$, $\bar\omega(K_i) = \omega(K_i) \subset U_i$. For $i \le k$,

$$\bar\omega(K_i) = \bar\omega(K_i \cap [0,t']) \cup \bar\omega(K_i \cap [t',1]) \subset \omega(K_i) \cup \omega'(I) \subset U_i \cup V = U_i$$

Therefore $\bar\omega \in W$ and $\varphi(\bar\omega) = y'$. Hence $\varphi(W) \supset V$. ∎

We can put these results together to obtain the following result, called the *lifting theorem.*

5 THEOREM *Let p: $(\tilde X,\tilde x_0) \to (X,x_0)$ be a fibration with unique path lifting. Let Y be a connected locally path-connected space. A necessary and sufficient condition that a map f: $(Y,y_0) \to (X,x_0)$ have a lifting $(Y,y_0) \to (\tilde X,\tilde x_0)$ is that in $\pi(X,x_0)$*

$$f_\#\pi(Y,y_0) \subset p_\#\pi(\tilde X,\tilde x_0)$$

PROOF If f': $(Y,y_0) \to (\tilde X,\tilde x_0)$ is a lifting of f, then $f = p \circ f'$ and

$$f_\#\pi(Y,y_0) = p_\# f'_\#\pi(Y,y_0) \subset p_\#\pi(\tilde X,\tilde x_0)$$

which shows that the condition is necessary.

We now prove that the condition is sufficient. It follows from lemmas 3 and 2 that if ω_0 is the constant path at y_0, the composite

$$(P(Y,y_0),\ \omega_0) \xrightarrow{\varphi} (Y,y_0) \xrightarrow{f} (X,x_0)$$

can be lifted to a map $\tilde f$: $(P(Y,y_0),\ \omega_0) \to (\tilde X,\tilde x_0)$. We show that if $f_\#\pi(Y,y_0) \subset p_\#\pi(\tilde X,\tilde x_0)$ and if $\omega, \omega' \in P(Y,y_0)$ are such that $\varphi(\omega) = \varphi(\omega')$, then $\tilde f(\omega) = \tilde f(\omega')$. Let $\bar\omega$ and $\bar\omega'$ be the paths in $P(Y,y_0)$ from ω_0 to ω and ω', respectively, defined by $\bar\omega(t)(t') = \omega(tt')$ and $\bar\omega'(t)(t') = \omega'(tt')$. Then $\tilde f \circ \bar\omega$ and $\tilde f \circ \bar\omega'$ are paths in $\tilde X$ from $\tilde x_0$ to $\tilde f(\omega)$ and $\tilde f(\omega')$, respectively, such that

$$p \circ \tilde f \circ \bar\omega = f \circ \varphi \circ \bar\omega = f \circ \omega \qquad \text{and} \qquad p \circ \tilde f \circ \bar\omega' = f \circ \omega'$$

Because $\omega * \omega'^{-1}$ is a closed path in Y at y_0 and $f_\#\pi(Y,y_0) \subset p_\#\pi(\tilde X,\tilde x_0)$, there is a closed path $\tilde\omega$ in $\tilde X$ at $\tilde x_0$ such that $(f \circ \omega) * (f \circ \omega')^{-1} \simeq p \circ \tilde\omega$. Then

$$p \circ (\tilde f \circ \bar\omega) = f \circ \omega \simeq (p \circ \tilde\omega) * (f \circ \omega') = p \circ (\tilde\omega * (\tilde f \circ \bar\omega'))$$

By lemma 2.3.3, $\tilde f \circ \bar\omega \simeq \tilde\omega * (\tilde f \circ \bar\omega')$. In particular, the endpoint of $\tilde f \circ \bar\omega$, which is $\tilde f(\omega)$, equals the endpoint of $\tilde f \circ \bar\omega'$, which is $\tilde f(\omega')$.

It follows that there is a function f': $(Y,y_0) \to (\tilde X,\tilde x_0)$ such that $f' \circ \varphi = \tilde f$, and using theorem 4, we see that f' is continuous. Because

$$p \circ f' \circ \varphi = p \circ \tilde{f} = f \circ \varphi$$

and φ is surjective, $p \circ f' = f$. Therefore f' is a lifting of f. ■

Let p: $E \to B$ be a fibration. A *section* of p is a map s: $B \to E$ such that $p \circ s = 1_B$ (thus a section is a right inverse of p). It follows easily from the homotopy lifting property that there is a section of p if and only if $[p]$ has a right inverse in the homotopy category. Because a section is a lifting of the identity map $B \subset B$, the following is an immediate consequence of theorem 5.

6 COROLLARY *Let p: $(\tilde{X},\tilde{x}_0) \to (X,x_0)$ be a fibration with unique path lifting. If X is a connected locally path-connected space, there is a section $(X,x_0) \to (\tilde{X},\tilde{x}_0)$ of p if and only if $p_\#\pi(\tilde{X},\tilde{x}_0) = \pi(X,x_0)$.* ■

7 COROLLARY *Let p: $\tilde{X} \to X$ be a fibration with unique path lifting. If $\tilde{X}$ is a nonempty path-connected space and X is connected and locally path connected, then p is a homeomorphism if and only if for some $\tilde{x}_0 \in \tilde{X}$, $p_\#\pi(\tilde{X},\tilde{x}_0) = \pi(X,p(\tilde{x}_0))$.*

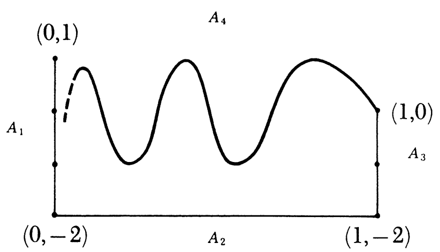
PROOF If p is a homeomorphism, $p_\#\pi(\tilde{X},\tilde{x}_0) = \pi(X,p(\tilde{x}_0))$. Conversely, if $p_\#\pi(\tilde{X},\tilde{x}_0) = \pi(X,p(\tilde{x}_0))$, then by theorem 2.3.9, p is a bijection. By corollary 6, it has a continuous right inverse. Therefore p is a homeomorphism. ■

If p: $\tilde{X} \to X$ is a covering projection and $\tilde{X}$ is path connected, a necessary and sufficient condition that p be a homeomorphism is that $p_\#\pi(\tilde{X},\tilde{x}_0) = \pi(X,p(\tilde{x}_0))$ for some $\tilde{x}_0 \in \tilde{X}$. This condition on the fundamental groups implies that p is a bijection, and by lemmas 2.1.8 and 2.1.7, p is open; hence for covering projections corollary 7 is valid without the assumption that X be locally path connected. This is definitely false for fibrations with unique path lifting if X is not locally path connected, because p need not be open. The following example shows this.

8 EXAMPLE Let X be the subspace of $\mathbf{R}^2$ defined to be the union of the four sets

$$\begin{aligned} A_1 &= \{(x,y) \mid x = 0,\ -2 \le y \le 1\} \\ A_2 &= \{(x,y) \mid 0 \le x \le 1,\ y = -2\} \\ A_3 &= \{(x,y) \mid x = 1,\ -2 \le y \le 0\} \\ A_4 &= \{(x,y) \mid 0 < x \le 1,\ y = \sin 2\pi/x\} \end{aligned}$$

illustrated in the diagram



Let $\tilde{X}$ be the half-open interval $[0,4)$ and define $p\colon \tilde{X} \to X$ to map $[0,1]$ linearly onto A_1, $[1,2]$ linearly onto A_2, $[2,3]$ linearly onto A_3, and $[3,4)$ homeomorphically onto A_4 by the map $t \to (t-3, \sin(2\pi/(t-3)))$. Then $\tilde{X}$ and X are path connected and $p\colon \tilde{X} \to X$ is a fibration with unique path lifting. However, p is not a homeomorphism, although $\tilde{X}$ and X are both simply connected.

For locally path-connected spaces the lifting theorem provides the following criterion for determining whether an open path-connected subset of the base space is evenly covered by a fibration.

9 LEMMA *Let $p\colon \tilde{X} \to X$ be a fibration with unique path lifting. Assume that $\tilde{X}$ and X are locally path connected and let U be an open connected subset of X. Then U is evenly covered by p if and only if every lifting to $\tilde{X}$ of a closed path in U is a closed path.*

PROOF If U is evenly covered by p and $\tilde{\omega}$ is a path in $p^{-1}(U)$, then $\tilde{\omega}$ is a path in some component $\tilde{U}$ of $p^{-1}(U)$. By lemma 2.1.12, p maps $\tilde{U}$ homeomorphically onto U. Therefore, if $p \circ \tilde{\omega}$ is a closed path in U, $\tilde{\omega}$ is a closed path in $\tilde{U}$. Hence the condition is necessary.

It is also sufficient, because if $x_0 \in U$ and $\tilde{x}_0 \in p^{-1}(x_0)$, the hypothesis that every lifting of a closed path in U at x_0 is a closed path in $\tilde{X}$ implies that in $\pi(X,x_0)$

$$i_{\#}\pi(U,x_0) \subset p_{\#}\pi(\tilde{X},\tilde{x}_0) \qquad \text{where } i\colon (U,x_0) \subset (X,x_0)$$

By theorem 5, there is a lifting $i'_{\tilde{x}_0}\colon (U,x_0) \to (\tilde{X},\tilde{x}_0)$ of i. The collection $\{i'_{\tilde{x}_0}(U) \mid \tilde{x}_0 \in p^{-1}(x_0)\}$ consists of path-connected sets which, by lemma 2.2.4, are disjoint. We show that their union equals $p^{-1}(U)$. If $\tilde{x} \in p^{-1}(U)$, let ω be a path in U from $p(\tilde{x})$ to x_0 and let $\tilde{\omega}$ be a lifting of ω such that $\tilde{\omega}(0) = \tilde{x}$. Then $\tilde{\omega}(1) \in p^{-1}(x_0)$, and therefore $\tilde{\omega}$ is a path in $i'_{\tilde{\omega}(1)}(U)$. Hence $\tilde{x} \in i'_{\tilde{\omega}(1)}(U)$ and $\{i'_{\tilde{x}_0}(U) \mid \tilde{x}_0 \in p^{-1}(x_0)\}$ is a partition of $p^{-1}(U)$ into path-connected sets. Since $p^{-1}(U)$ is open and $\tilde{X}$ is locally path connected, $i'_{\tilde{x}_0}(U)$ is open in $\tilde{X}$ for each $\tilde{x}_0 \in p^{-1}(x_0)$. Clearly, p is a homeomorphism of $i'_{\tilde{x}_0}(U)$ onto U for each $\tilde{x}_0 \in p^{-1}(x_0)$, and U is evenly covered by p. ∎

A space X is said to be *semilocally 1-connected* if every point $x_0 \in X$ has a neighborhood N such that $\pi(N,x_0) \to \pi(X,x_0)$ is trivial.

10 THEOREM *Every fibration with unique path lifting whose base space is locally path connected and semilocally 1-connected and whose total space is locally path connected is a covering projection.*

PROOF It follows from lemma 9 and the definition of semilocally 1-connected space that each point of the base space has an open neighborhood evenly covered by the fibration. ∎

5 THE CLASSIFICATION OF COVERING PROJECTIONS

This section contains a classification of covering projections over a connected locally path-connected base space. It is based on the lifting theorem and reduces the problem of equivalence of covering projections to conjugacy of their corresponding subgroups of the fundamental group of the base space. A large part of the section is devoted to constructing a covering projection corresponding to a given subgroup of the fundamental group of the base space.

Let X be a connected space. The *category of connected covering spaces* of X has objects which are covering projections $p\colon \tilde{X} \to X$, where $\tilde{X}$ is connected, and morphisms which are commutative triangles

$$\begin{array}{ccc} \tilde{X}_1 & \xrightarrow{f} & \tilde{X}_2 \\ & {\scriptstyle p_1}\searrow \quad \swarrow{\scriptstyle p_2} & \\ & X & \end{array}$$

If X is locally path connected and $p\colon \tilde{X} \to X$ is an object of this category, then, by lemma 2.1.8, p is a local homeomorphism and $\tilde{X}$ is also locally path connected. We show that in this case every morphism in this category is a covering projection.

1 LEMMA *In the category of connected covering spaces of a connected locally path-connected space every morphism is itself a covering projection.*

PROOF Consider a commutative triangle

$$\begin{array}{ccc} \tilde{X}_1 & \xrightarrow{f} & \tilde{X}_2 \\ & {\scriptstyle p_1}\searrow \quad \swarrow{\scriptstyle p_2} & \\ & X & \end{array}$$

where p_1 and p_2 are covering projections and X is locally path connected. It follows from corollary 2.1.13 that f is a covering projection if it is surjective.

Because $\tilde{X}_2$ is connected and locally path connected, it is path connected. Let $\tilde{x}_1 \in \tilde{X}_1$ and $\tilde{x}_2 \in \tilde{X}_2$ be arbitrary and let $\tilde{\omega}_2$ be a path in $\tilde{X}_2$ from $f(\tilde{x}_1)$ to $\tilde{x}_2$. Because p_1 is a fibration, there is a path $\tilde{\omega}_1$ in $\tilde{X}_1$ beginning at $\tilde{x}_1$ such that $p_1 \circ \tilde{\omega}_1 = p_2 \circ \tilde{\omega}_2$. By the unique path lifting of p_2, $f \circ \tilde{\omega}_1 = \tilde{\omega}_2$. Therefore

$$f(\tilde{\omega}_1(1)) = \tilde{\omega}_2(1) = \tilde{x}_2$$

proving that f is surjective. ∎

The next result determines when there is a morphism from one object to another in the category of connected covering spaces of X.

2 THEOREM *Let $p_1\colon \tilde{X}_1 \to X$ and $p_2\colon \tilde{X}_2 \to X$ be objects in the category* *of connected covering spaces of a connected locally path-connected space X. The following are equivalent:*

(*a*) *There is a covering projection $f\colon \tilde{X}_1 \to \tilde{X}_2$ such that $p_2 \circ f = p_1$.*
(*b*) *For all $\tilde{x}_1 \in \tilde{X}_1$ and $\tilde{x}_2 \in \tilde{X}_2$ such that $p_1(\tilde{x}_1) = p_2(\tilde{x}_2)$, $p_{1\#}\pi(\tilde{X}_1,\tilde{x}_1)$ is conjugate in $\pi(X,p_1(\tilde{x}_1))$ to a subgroup of $p_{2\#}\pi(\tilde{X}_2,\tilde{x}_2)$.*
(*c*) *There exist $\tilde{x}_1 \in \tilde{X}_1$ and $\tilde{x}_2 \in \tilde{X}_2$ such that $p_1(\tilde{x}_1) = p_2(\tilde{x}_2)$ and $p_{1\#}\pi(\tilde{X}_1,\tilde{x}_1)$ is conjugate in $\pi(X,p_1(\tilde{x}_1))$ to a subgroup of $p_{2\#}\pi(\tilde{X}_2,\tilde{x}_2)$.*

PROOF (*a*) $\Rightarrow$ (*b*) Given $f\colon \tilde{X}_1 \to \tilde{X}_2$ such that $p_2 \circ f = p_1$, if $\tilde{x}_1 \in \tilde{X}_1$ and $\tilde{x}_2 \in \tilde{X}_2$ are such that $p_1(\tilde{x}_1) = p_2(\tilde{x}_2)$, then

$$p_{1\#}\pi(\tilde{X}_1,\tilde{x}_1) = p_{2\#} \circ f_{\#}\pi(\tilde{X}_1,\tilde{x}_1) \subset p_{2\#}\pi(\tilde{X}_2,f(\tilde{x}_1))$$

Because $f(\tilde{x}_1)$ and $\tilde{x}_2$ lie in the same fiber of $p_2\colon \tilde{X}_2 \to X$, it follows from theorem 2.3.6 that $p_{2\#}\pi(\tilde{X}_2,f(\tilde{x}_1))$ and $p_{2\#}\pi(\tilde{X}_2,\tilde{x}_2)$ are conjugate in $\pi(X,p_1(\tilde{x}_1))$.

(*b*) $\Rightarrow$ (*c*) The proof is trivial.

(*c*) $\Rightarrow$ (*a*) Assume that $\tilde{x}_1 \in \tilde{X}_1$ and $\tilde{x}_2 \in \tilde{X}_2$ are such that $p_1(\tilde{x}_1) = p_2(\tilde{x}_2)$ and that $p_{1\#}\pi(\tilde{X}_1,\tilde{x}_1)$ is conjugate in $\pi(X,p_1(\tilde{x}_1))$ to a subgroup of $p_{2\#}\pi(\tilde{X}_2,\tilde{x}_2)$. By theorem 2.3.6, there is a point $\tilde{x}_2' \in \tilde{X}_2$ such that $p_2(\tilde{x}_2') = p_2(\tilde{x}_2)$ and such that $p_{1\#}\pi(\tilde{X}_1,\tilde{x}_1) \subset p_{2\#}\pi(\tilde{X}_2,\tilde{x}_2')$

Because $\tilde{X}_1$ is a connected locally path-connected space, the lifting theorem implies the existence of a map $f\colon (\tilde{X}_1,\tilde{x}_1) \to (\tilde{X}_2,\tilde{x}_2')$ such that $p_2 \circ f = p_1$. ∎

3 COROLLARY *Two objects in the category of connected covering spaces of a connected locally path-connected space X are equivalent if and only if their fundamental groups (at some two points over the same point of X) map to conjugate subgroups of the fundamental group of X (at this point).* ∎

We give two examples.

4 Because every nontrivial subgroup of $\pi(S^1) \approx \mathbf{Z}$ is infinite cyclic, by corollary 3 every connected covering space $\tilde{X} \to S^1$ is equivalent to $ex\colon \mathbf{R} \to S^1$ or to the map $S^1 \to S^1$ sending z to z^n for some positive integer n.

5 For $n \geq 2$, $\pi(P^n) \approx \mathbf{Z}_2$, and every connected covering space $\tilde{X} \to P^n$ is equivalent to the double covering $S^n \to P^n$ or to the trivial covering $P^n \subset P^n$.

A *universal covering space* of a connected space X is an object $p\colon \tilde{X} \to X$ of the category of connected covering spaces of X such that for any object $p'\colon \tilde{X}' \to X$ of this category there is a morphism

$$\begin{tikzcd} \tilde{X} \arrow[rr, "f"] \arrow[dr, "p"'] & & \tilde{X}' \arrow[dl, "p'"] \\ & X & \end{tikzcd}$$

in the category. It can be shown (see the paragraph following theorem 13 below) that a universal covering space is a regular covering space. The next result follows from this, theorem 2 and corollary 3.

6 COROLLARY *Two universal covering spaces of a connected locally path-connected space are equivalent.* ∎

Another result also follows from theorem 2.

7 COROLLARY *A simply connected covering space of a connected locally path-connected space X is a universal covering space of X.* ■

Having reduced the comparison of connected covering spaces of X to a comparison of their corresponding subgroups of the fundamental group of X, we shall determine which subgroups of the fundamental group correspond to covering spaces. This necessitates the construction of covering spaces. Let X be a space and let $\mathfrak{U}$ be an open covering of X. If $x_0 \in X$, let $\pi(\mathfrak{U},x_0)$ be the subgroup of $\pi(X,x_0)$ generated by homotopy classes of closed paths having a representative of the form $(\omega * \omega') * \omega^{-1}$, where ω' is a closed path lying in some element of $\mathfrak{U}$ and ω is a path from x_0 to $\omega'(0)$. The following statements are easily verified.

8 *If $\mathcal{V}$ is an open covering of X that refines $\mathfrak{U}$, then $\pi(\mathcal{V},x_0) \subset \pi(\mathfrak{U},x_0)$.* ■

9 *$\pi(\mathfrak{U},x_0)$ is a normal subgroup of $\pi(X,x_0)$.* ■

10 *If ω is a path in X, then $h_{[\omega]}\pi(\mathfrak{U},\omega(1)) = \pi(\mathfrak{U},\omega(0))$.* ■

The connection of the groups $\pi(\mathfrak{U},x_0)$ with covering projections is explained by the following result.

11 LEMMA *Let $p\colon \tilde{X} \to X$ be a covering projection and let $\mathfrak{U}$ be a covering of X by open sets each evenly covered by p. For any $\tilde{x}_0 \in \tilde{X}$*

$$\pi(\mathfrak{U},p(\tilde{x}_0)) \subset p_\# \pi(\tilde{X},\tilde{x}_0)$$

PROOF If ω' is a closed path lying in some element of $\mathfrak{U}$, then, by lemma 2.4.9, any lifting of ω' is a closed path in $\tilde{X}$. Hence any path of the form $(\omega * \omega') * \omega^{-1}$, where ω' is a closed path lying in some element of $\mathfrak{U}$, can be lifted to a closed path [namely, to $(\tilde{\omega} * \tilde{\omega}') * \tilde{\omega}^{-1}$, where $\tilde{\omega}$ and $\tilde{\omega}'$ are suitable liftings of ω and ω', respectively]. Hence any element of $\pi(\mathfrak{U},p(\tilde{x}_0))$ has a representative which can be lifted to a closed path at $\tilde{x}_0$. ■

The following theorem characterizes those fibrations with unique path lifting which are covering projections.

12 THEOREM *Let $p\colon \tilde{X} \to X$ be a fibration with unique path lifting, where X and $\tilde{X}$ are connected locally path-connected spaces. Then p is a covering projection if and only if there is an open covering $\mathfrak{U}$ of X and a point $\tilde{x}_0 \in \tilde{X}$ such that*

$$\pi(\mathfrak{U},p(\tilde{x}_0)) \subset p_\# \pi(\tilde{X},\tilde{x}_0)$$

PROOF If p is a covering projection, the desired result follows from lemma 11. Conversely, if there is such an open covering $\mathfrak{U}$ and point $\tilde{x}_0 \in \tilde{X}$, it follows from statements 9 and 10 that for any point $\tilde{x}_0' \in \tilde{X}$, $\pi(\mathfrak{U},p(\tilde{x}_0')) \subset p_\# \pi(\tilde{X},\tilde{x}_0')$. Using lemma 2.4.9, it follows that every element of $\mathfrak{U}$ is evenly covered by p. ■

Lemma 11 gives a necessary condition for a subgroup of $\pi(X,x_0)$ to correspond to a covering space. The next result proves that this necessary condition is also sufficient.

13 THEOREM *Let X be a connected locally path-connected space and let $x_0 \in X$. Let H be a subgroup of $\pi(X,x_0)$ and assume that there is an open covering $\mathcal{U}$ of X such that $\pi(\mathcal{U},x_0) \subset H$. Then there is a covering projection $p\colon (\tilde{X},\tilde{x}_0) \to (X,x_0)$ such that $p_\#\pi(\tilde{X},\tilde{x}_0) = H$.*

PROOF Suppose such a covering projection exists, and suppose, moreover, that the space $\tilde{X}$ is path connected. The projection $\varphi\colon (P(X,x_0),\omega_0) \to (X,x_0)$ of the path space of (X,x_0) can then be lifted to a map $\varphi'\colon (P(X,x_0),\omega_0) \to (\tilde{X},\tilde{x}_0)$, which is surjective. If ω and ω' are elements of $P(X,x_0)$, then $\varphi'(\omega) = \varphi'(\omega')$ if and only if $\varphi(\omega) = \varphi(\omega')$ and $[\omega * \omega'^{-1}] \in p_\#\pi(\tilde{X},\tilde{x}_0) = H$. Therefore, for path connected $\tilde{X}$ there is a one-to-one correspondence between the points of $\tilde{X}$ and equivalence classes of $P(X,x_0)$ identifying ω with ω' if $\omega(1) = \omega'(1)$ and $[\omega * \omega'^{-1}] \in H$ (the group properties of H imply that this is an equivalence relation). Hence it is natural to try to construct $\tilde{X}$ by suitably topologizing these equivalence classes of $P(X,x_0)$. We could start with the compact-open topology on $P(X,x_0)$ and use the quotient topology on the set of equivalence classes, but it seems no simpler than merely topologizing the set of equivalence classes directly, as is done below.

We consider the set of all paths in X beginning at x_0. If ω and ω' are two such paths, set $\omega \sim \omega'$ if $\omega(1) = \omega'(1)$ and $[\omega * \omega'^{-1}] \in H$. This is an equivalence relation, and the equivalence class of ω will be denoted by $\langle\omega\rangle$. Let $\tilde{X}$ be the set of equivalence classes. There is a function $p\colon \tilde{X} \to X$ such that $p(\langle\omega\rangle) = \omega(1)$. If U is an open subset of X and ω is a path beginning at x_0 and ending in U, $\langle\omega,U\rangle$ will denote the subset of $\tilde{X}$ consisting of all the equivalence classes having a representative of the form $\omega * \omega'$, where ω' is a path in U beginning at $\omega(1)$.

We prove that the collection $\{\langle\omega,U\rangle\}$ is a base for a topology on $\tilde{X}$. If $\langle\omega'\rangle \in \langle\omega,U\rangle$, then $\omega' \sim \omega * \omega''$ for some path ω'' lying in U. If $\bar{\omega}$ is any path in U beginning at $\omega'(1)$, then

$$\omega' * \bar{\omega} \sim (\omega * \omega'') * \bar{\omega} \sim \omega * (\omega'' * \bar{\omega})$$

showing that $\langle\omega',U\rangle \subset \langle\omega,U\rangle$. Since $\omega \sim \omega' * \omega''^{-1}$, $\langle\omega\rangle \in \langle\omega',U\rangle$. The same argument shows that $\langle\omega,U\rangle \subset \langle\omega',U\rangle$, and so $\langle\omega,U\rangle = \langle\omega',U\rangle$. Therefore, if $\omega'' \in \langle\omega,U\rangle \cap \langle\omega',U'\rangle$, then $\langle\omega'', U \cap U'\rangle \subset \langle\omega,U\rangle \cap \langle\omega',U'\rangle$, and so the collection $\{\langle\omega,U\rangle\}$ is a base for a topology on $\tilde{X}$.

Let $\tilde{X}$ be topologized by the topology having $\{\langle\omega,U\rangle\}$ as a base. Then p is continuous; for if $p(\langle\omega\rangle) \in U$, then $p(\langle\omega,U\rangle) \subset U$. p is also open, because $p(\langle\omega,U\rangle)$ clearly equals the path component of U containing $\omega(1)$, and this is open because X is locally path connected.

Let $\mathcal{U}$ be an open covering of X such that $\pi(\mathcal{U},x_0) \subset H$ and let V be an open path-connected subset of X contained in some element of $\mathcal{U}$. We show that V is evenly covered by p, which will imply that p is a covering projection.

If $\langle\omega\rangle \in p^{-1}(V)$, then $\langle\omega,V\rangle \subset p^{-1}(V)$. The sets $\{\langle\omega,V\rangle \mid \langle\omega\rangle \in p^{-1}(V)\}$ are open and their union equals $p^{-1}(V)$. If $\langle\omega,V\rangle \cap \langle\omega',V\rangle \neq \varnothing$, let $\langle\omega''\rangle \in \langle\omega,V\rangle \cap \langle\omega',V\rangle$. Then $\langle\omega'',V\rangle = \langle\omega,V\rangle$ and $\langle\omega'',V\rangle = \langle\omega',V\rangle$. Hence the sets $\{\langle\omega,V\rangle \mid \langle\omega\rangle \in p^{-1}(V)\}$ are either identical or disjoint. To prove that V is evenly covered by p, it suffices to show that p maps each set $\langle\omega,V\rangle$ bijectively to V (because p has already been shown to be continuous and open). If $x \in V$, let ω' be a path in V from $\omega(1)$ to x. Then $\langle\omega * \omega'\rangle \in \langle\omega,V\rangle$ and $p(\langle\omega * \omega'\rangle) = x$, showing that p is surjective. Assume $p\langle\omega * \omega_1\rangle = p\langle\omega * \omega_2\rangle$. Then $\omega_1(1) = \omega_2(1)$, so $(\omega * \omega_1) * (\omega * \omega_2)^{-1}$ is a closed path in X at x_0. Also, $[(\omega * \omega_1) * (\omega * \omega_2)^{-1}] = [(\omega * (\omega_1 * \omega_2{}^{-1})) * \omega^{-1}]$
Since $\omega_1 * \omega_2{}^{-1}$ is a path in V and V is contained in some element of U, $[(\omega * (\omega_1 * \omega_2{}^{-1})) * \omega^{-1}] \in \pi(\mathfrak{U},x_0) \subset H$. Therefore $\omega * \omega_1 \sim \omega * \omega_2$ and $\langle\omega * \omega_1\rangle = \langle\omega * \omega_2\rangle$, showing that p is injective.

We have shown that $p\colon \tilde{X} \to X$ is a covering projection. Let $\tilde{x}_0 = \langle\omega_0\rangle$, where ω_0 is the constant path in X at x_0. It remains only to verify that $p_\#\pi(\tilde{X},\tilde{x}_0) = H$. For this we need an explicit expression for the lift of a path in X that begins at x_0. Let ω be a path in X beginning at x_0, and for $t \in I$, define a path ω_t in X beginning at x_0 by $\omega_t(t') = \omega(tt')$. Let $\tilde{\omega}\colon I \to \tilde{X}$ be defined by $\tilde{\omega}(t) = \langle\omega_t\rangle$. We prove that $\tilde{\omega}$ is continuous. If $\tilde{\omega}(t_0) \in \langle\omega',U\rangle$, then $p\tilde{\omega}(t_0) = \omega(t_0) \in U$ and $\langle\omega',U\rangle = \langle\omega_{t_0},U\rangle$. Let N be any open interval in I containing t_0 such that $\omega(N) \subset U$. If $t \in N$, then $\omega_t \sim \omega_{t_0} * \omega_{t_0,t}$, where $\omega_{t_0,t}(t') = \omega(t_0 + t'(t - t_0))$. Therefore, for $t \in N$

$$\tilde{\omega}(t) = \langle\omega_t\rangle = \langle\omega_{t_0} * \omega_{t_0,t}\rangle \in \langle\omega_{t_0},U\rangle = \langle\omega',U\rangle$$

and so $\tilde{\omega}$ is continuous. Furthermore, $p\tilde{\omega}(t) = \omega_t(1) = \omega(t)$. Hence $\tilde{\omega}$ is a lift of ω beginning at $\tilde{\omega}(0) = \tilde{x}_0$ and ending at $\tilde{\omega}(1) = \langle\omega\rangle$.

If $[\omega] \in H$, then $\omega \sim \omega_0$ and $\langle\omega\rangle = \tilde{x}_0$. Therefore the lift $\tilde{\omega}$ of ω constructed above is a closed path in $\tilde{X}$ at $\tilde{x}_0$, proving that $H \subset p_\#\pi(\tilde{X},\tilde{x}_0)$. On the other hand, if $\tilde{\omega}'$ is a closed path in $\tilde{X}$ at $\tilde{x}_0$ and $p\tilde{\omega}' = \omega$, let $\tilde{\omega}$ be the path in $\tilde{X}$ constructed above. Since $\tilde{\omega}$ is a lift of ω beginning at $\tilde{x}_0$, it follows from the unique path lifting of p that $\tilde{\omega} = \tilde{\omega}'$. Therefore $\tilde{\omega}(1) = \tilde{\omega}'(1) = \tilde{x}_0$. Since $\tilde{\omega}(1) = \langle\omega\rangle$, $\omega \sim \omega_0$, showing that $p_\#\pi(\tilde{X},\tilde{x}_0) \subset H$. ∎
Note that if $p\colon \tilde{X} \to X$ is a universal covering space it is a regular covering.
In fact, if $\tilde{x}_0 \in \tilde{X}$ and $\mathscr{U}$ is a covering of X by open sets evenly covered by p than by 2.5.11 $\pi(\mathscr{U}, p(\tilde{x}_0)) \subset p_\#\pi(\tilde{X},\tilde{x}_0) \subset \pi(X, p(\tilde{x}_0))$
By 2.5.13 there exists a connected covering $q\colon (\tilde{Y},\tilde{y}) \to (X, p(\tilde{x}_0))$ such that $q_\#\pi(\tilde{Y},\tilde{y}) = \pi(\mathscr{U}, p(\tilde{x}_0))$. Since $p\colon \tilde{X} \to X$ is universal there is a map $f\colon \tilde{X} \to \tilde{Y}$ such that $qf = p$. By 2.5.2, $p_\#\pi(\tilde{X},\tilde{x}_0)$ is conjugate in $\pi(X,p(\tilde{x}_0))$ to a subgroup of $\pi(\mathscr{U}, p(\tilde{x}_0))$. By 2.5.9, $\pi(\mathscr{U}, p(\tilde{x}_0))$ is normal so we must have $p_\#\pi(\tilde{X},\tilde{x}_0) \subset \pi(\mathscr{U}, p(\tilde{x}_0)$ and so $p_\#\pi(\tilde{X},\tilde{x}_0) = \pi(\mathscr{U}, p(\tilde{x}_0))$ is normal.

A space X is semilocally 1-connected (defined in Sec. 2.4) if and only if there is an open covering $\mathfrak{U}$ of X such that $\pi(\mathfrak{U},x_0) = 0$. Hence we have the following result.

14 COROLLARY *A connected locally path-connected space X has a simply connected covering space if and only if X is semilocally 1-connected.* ∎

From corollaries 14 and 6 and theorem 2 we obtain the next result.

15 COROLLARY *Any universal covering space of a connected locally path-connected semilocally 1-connected space is simply connected.* ∎

Not every connected locally path-connected space has a universal covering space. We give two examples.

16 An infinite product of 1-spheres has no universal covering space.

17 Let X be the subspace of $\mathbf{R}^2$ equal to the union of the circumferences of circles C_n, with $n \geq 1$, where C_n has center at $(1/n, 0)$ and radius $1/n$. Then X is connected and locally path connected but has no universal covering space.

It is possible for a connected locally path-connected space to have a universal covering space that is not simply connected. We present an example.

18 EXAMPLE Let Y_1 be the cone with base X equal to the space of example 17 [Y_1 can be visualized as the set of line segments in $\mathbf{R}^3$ joining the points of X to the point $(0,0,1)$] and let y_1 be the point at which all the circles of X are tangent. Let (Y_2,y_2) be another copy of (Y_1,y_1). Let $Z = Y_1 \vee Y_2$. Then Z is connected and locally path connected but not simply connected (cf. exercise 1.G.7, a closed path oscillating back and forth from Y_1 to Y_2 around the decreasing circles C_n is not null homotopic). However, Y_1 and Y_2 are each closed contractible subsets of Z. By the lifting theorem, each of them can be lifted to any covering space of Z, so that y_1 is lifted arbitrarily and y_2 is lifted arbitrarily. Therefore any covering projection with base Z has a section. It follows that any connected covering space of Z is homeomorphic to Z.

In the category of fibrations with unique path lifting over a fixed path-connected base space (and with path-connected total spaces) there is always a universal object (that is, an object which has morphisms to any other object in the category). We sketch a proof of this fact. Let X be a path-connected space and let $\mathfrak{X}(X)$ be the collection of topological spaces whose underlying sets are cartesian products of X and the set of right cosets of some subgroup of the fundamental group of X. It follows from theorem 2.3.9 that any fibration whose base space is X and total space is path connected is equivalent to a fibration $\tilde{X} \to X$, where $\tilde{X} \in \mathfrak{X}(X)$. Since $\mathfrak{X}(X)$ is a set, those fibrations $\tilde{X} \to X$ with unique path lifting, where $\tilde{X}$ is a path-connected space in $\mathfrak{X}(X)$, constitute a set. We may form the fibered product of this set (as in Sec. 2.2). Any path component of this fibered product is then the desired universal fibration with unique path lifting.

If X is a connected locally path-connected space, it follows from theorem 13 that for any open covering $\mathfrak{U}$ of X there is a path-connected covering space of X whose fundamental group is isomorphic to $\pi(\mathfrak{U},x_0)$. This implies that if $\tilde{X}$ is a universal object in the category of path-connected fibrations over X with unique path lifting, then $\pi(\tilde{X},\tilde{x}_0)$ is isomorphic to a subgroup of $\bigcap_{\mathfrak{U}} \pi(\mathfrak{U},x_0)$. In particular, if $\bigcap_{\mathfrak{U}} \pi(\mathfrak{U},x_0) = 0$, then X has a simply connected fibration with unique path lifting that is a universal object in the category. Thus the spaces in examples 16 and 17 both have universal fibrations with unique path lifting that are simply connected. The space Z of example 18 is its own universal fibration with unique path lifting.

6 COVERING TRANSFORMATIONS

In this section we consider a problem inverse to the one of the last section, in which we constructed covering projections with given base space; we ask for covering projections with given covering space. On any regular covering space we prove that there is a group of covering transformations. The covering projection is then equivalent to the projection of the covering space onto the space of orbits of the group of covering transformations.

Let $p\colon \tilde{X} \to X$ be a fibration with unique path lifting. It is clear that there is a *group of self-equivalences* of this fibration (a self-equivalence is a homeomorphism $f\colon \tilde{X} \to \tilde{X}$ such that $p \circ f = p$). We denote this group by $G(\tilde{X} \mid X)$. In case $p\colon \tilde{X} \to X$ is a covering projection, $G(\tilde{X} \mid X)$ is also called the *group of covering transformations* of p. In general, there is a close analogy of $G(\tilde{X} \mid X)$ with the group of automorphisms of an extension field leaving a subfield pointwise fixed.

If $\tilde{X}$ is path connected, it follows from lemma 2.2.4 that two self-equivalences of $p\colon \tilde{X} \to X$ that agree at one point are identical. Hence we have the following lemma.

1 LEMMA *Let $p\colon \tilde{X} \to X$ be a fibration with unique path lifting. If $\tilde{X}$ is path connected and $\tilde{x}_0 \in \tilde{X}$, then the function $f \to f(\tilde{x}_0)$ is an injection of $G(\tilde{X} \mid X)$ into the fiber of p over $p(\tilde{x}_0)$.* ∎

Theorem 2.3.9 established a bijection from the set of right cosets of $p_\#\pi(\tilde{X},\tilde{x}_0)$ in $\pi(X,p(\tilde{x}_0))$ to the fiber of p over $p(\tilde{x}_0)$. Combining the inverse of this bijection with the function of lemma 1 yields an injection ψ from $G(\tilde{X} \mid X)$ to the set of right cosets of $p_\#\pi(\tilde{X},\tilde{x}_0)$ in $\pi(X,p(\tilde{x}_0))$. ψ is defined explicitly as follows. For any $f \in G(\tilde{X} \mid X)$ let $\tilde{\omega}$ be a path in $\tilde{X}$ from $\tilde{x}_0$ to $f(\tilde{x}_0)$. Then $p \circ \tilde{\omega}$ is a closed path in X at $p(\tilde{x}_0)$, and the right coset $(p_\#\pi(\tilde{X},\tilde{x}_0))\,[p \circ \tilde{\omega}]$ is independent of the choice of $\tilde{\omega}$. The function ψ assigns to f this right coset.

Given $\tilde{x}_0 \in \tilde{X}$, let $N(p_\#\pi(\tilde{X},\tilde{x}_0))$ be the normalizer of $p_\#\pi(\tilde{X},\tilde{x}_0)$ in $\pi(X,p(\tilde{x}_0))$. Thus $N(p_\#\pi(\tilde{X},\tilde{x}_0))$ is the subgroup of $\pi(X,p(\tilde{x}_0))$ consisting of elements $[\omega] \in \pi(X,p(\tilde{x}_0))$ such that $p_\#\pi(\tilde{X},\tilde{x}_0)$ is invariant under conjugation by $[\omega]$. $N(p_\#\pi(\tilde{X},\tilde{x}_0))$ is the largest subgroup of $\pi(X,p(\tilde{x}_0))$ containing $p_\#\pi(\tilde{X},\tilde{x}_0)$ as a normal subgroup.

2 THEOREM *Let $p\colon \tilde{X} \to X$ be a fibration with unique path lifting. Let $\tilde{X}$ be path connected and let $\tilde{x}_0 \in \tilde{X}$. Then ψ is a monomorphism of $G(\tilde{X} \mid X)$ to the quotient group $N(p_\#\pi(\tilde{X},\tilde{x}_0))/p_\#\pi(\tilde{X},\tilde{x}_0)$. If $\tilde{X}$ is also locally path connected, ψ is an isomorphism.*

PROOF We already know that ψ is an injection. We show that ψ is a function from $G(\tilde{X} \mid X)$ to the set of right cosets of $p_\#\pi(\tilde{X},\tilde{x}_0)$ by elements of $N(p_\#\pi(\tilde{X},\tilde{x}_0))$.

If $\tilde{\omega}$ is a path in $\tilde{X}$ from $\tilde{x}_0$ to $f(\tilde{x}_0)$, there is a commutative square

$$\begin{tikzcd}
\pi(\tilde{X},\tilde{x}_0) \arrow[d, "p_\#"'] & \pi(\tilde{X},f(\tilde{x}_0)) \arrow[l, "h_{[\tilde{\omega}]}"'] \arrow[d, "p_\#"] \\
\pi(X,p(\tilde{x}_0)) & \pi(X,p(\tilde{x}_0)) \arrow[l, "h_{[p\circ\tilde{\omega}]}"']
\end{tikzcd}$$

Since $f\colon (\tilde{X},\tilde{x}_0) \to (\tilde{X},f(\tilde{x}_0))$ is a homeomorphism,

$$f_\#\pi(\tilde{X},\tilde{x}_0) = \pi(\tilde{X},f(\tilde{x}_0))$$

and since $p_\# f_\# = p_\#$,

$$\begin{aligned} h_{[p\circ\tilde{\omega}]}p_\#\pi(\tilde{X},\tilde{x}_0) &= h_{[p\circ\tilde{\omega}]}p_\# f_\#\pi(\tilde{X},\tilde{x}_0) = h_{[p\circ\tilde{\omega}]}p_\#\pi(\tilde{X},f(\tilde{x}_0)) \\ &= p_\# h_{[\tilde{\omega}]}\pi(\tilde{X},f(\tilde{x}_0)) \quad = p_\#\pi(\tilde{X},f(\tilde{x}_0)) \end{aligned}$$

Hence $[p \circ \tilde{\omega}] \in N(p_\#\pi(\tilde{X},\tilde{x}_0))$. Because $\psi(f)$ is equal to the right coset $(p_\#\pi(\tilde{X},\tilde{x}_0))\,[p \circ \tilde{\omega}]$, ψ is an injection of $G(\tilde{X} \mid X)$ into the set of right cosets of $p_\#\pi(\tilde{X},\tilde{x}_0)$ by elements of $N(p_\#\pi(\tilde{X},\tilde{x}_0))$.

We now verify that ψ is an homomorphism. If $f_1, f_2 \in G(\tilde{X} \mid X)$ let $\tilde{\omega}_1$ and $\tilde{\omega}_2$ be paths in $\tilde{X}$ from $\tilde{x}_0$ to $f_1(\tilde{x}_0)$ and $f_2(\tilde{x}_0)$, respectively. Then $f_1 \circ \tilde{\omega}_2$ is a path from $f_1(\tilde{x}_0)$ to $f_1f_2(\tilde{x}_0)$, and $\tilde{\omega}_1 * (f_1 \circ \tilde{\omega}_2)$ is a path from $\tilde{x}_0$ to $f_1f_2(\tilde{x}_0)$. Therefore $\psi(f_1f_2)$ is the right coset

$$(p_\#\pi(\tilde{X},\tilde{x}_0))[(p \circ \tilde{\omega}_1) * (p \circ f_1 \circ \tilde{\omega}_2)] = (p_\#\pi(\tilde{X},\tilde{x}_0))[p \circ \tilde{\omega}_1] * [p \circ \tilde{\omega}_2]$$

and this equals $\psi(f_1)\psi(f_2)$.

Finally, we show that if $\tilde{X}$ is locally path connected, ψ is an epimorphism to the set of right cosets of $p_\#\pi(\tilde{X},\tilde{x}_0)$ in $N(p_\#\pi(\tilde{X},\tilde{x}_0))$. Assume that $[\omega] \in \pi(X,p(\tilde{x}_0))$ belongs to $N(p_\#\pi(\tilde{X},\tilde{x}_0))$. Let $\tilde{\omega}$ be a lifting of ω ending at $\tilde{x}_0$ and let $\tilde{x} = \tilde{\omega}(0)$. Then

$$p_\#\pi(\tilde{X},\tilde{x}_0) = h_{[\omega]}(p_\#\pi(\tilde{X},\tilde{x}_0)) = p_\#(h_{[\tilde{\omega}]}\pi(\tilde{X},\tilde{x}_0)) = p_\#\pi(\tilde{X},\tilde{x})$$

Because X is connected and locally path connected, the lifting theorem implies the existence of maps $f\colon (\tilde{X},\tilde{x}_0) \to (\tilde{X},\tilde{x})$ and $g\colon (\tilde{X},\tilde{x}) \to (\tilde{X},\tilde{x}_0)$ such that $p \circ f = p$ and $p \circ g = p$. From the unique-lifting property (lemma 2.2.4), it follows that $f \circ g = 1_{\tilde{X}}$ and $g \circ f = 1_{\tilde{X}}$. Therefore $f \in G(\tilde{X} \mid X)$ and $\psi(f)$ equals the right coset $(p_\#\pi(\tilde{X},\tilde{x}_0))[\omega]^{-1}$. ∎

Combining theorem 2 with theorem 2.3.12, we have the following corollary.

3 COROLLARY *Let $p\colon \tilde{X} \to X$ be a fibration with unique path lifting. If $\tilde{X}$ is connected and locally path connected and $\tilde{x}_0 \in \tilde{X}$, then p is regular if and only if $G(\tilde{X} \mid X)$ is transitive on each fiber of p, in which case*

$$\psi\colon G(\tilde{X} \mid X) \approx \pi(X,p(\tilde{x}_0))/p_\#\pi(\tilde{X},\tilde{x}_0) \quad ∎$$

If $\tilde{X}$ is simply connected, any fibration $p: \tilde{X} \to X$ is regular, and we also have the next result.

4 COROLLARY *Let $p: \tilde{X} \to X$ be a fibration with unique path lifting, where $\tilde{X}$ is simply connected, locally path connected, and nonempty. Then the group of self-equivalences of p is isomorphic to the fundamental group of X.* ■

If $p: \tilde{X} \to X$ is a regular covering projection and $\tilde{X}$ is connected and locally path connected, then X is homeomorphic to the space of orbits of $G(\tilde{X} \mid X)$ (an *orbit* of a group of transformations G acting on a set S is an equivalence class of S with respect to the equivalence relation $s_1 \sim s_2$ if there is $g \in G$ such that $gs_1 = s_2$). We are interested in the converse problem —that is, in knowing what conditions on a group G of homeomorphisms of a topological space Y will ensure that the projection of Y onto the space of orbits Y/G is a regular covering projection whose group of covering transformations is equal to G.

A group G of homeomorphisms of a topological space Y is said to be *discontinuous* if the orbits of G in Y are discrete subsets of Y. G is *properly discontinuous* if for $y \in Y$ there is an open neighborhood U of y in Y such that if $g, g' \in G$ and gU meets $g'U$, then $g = g'$. *G acts without fixed points* if the only element of G having fixed points is the identity element. The following are clear.

5 *A properly discontinuous group of homeomorphisms is discontinuous and acts without fixed points.* ■

6 *A finite group of homeomorphisms acting without fixed points on a Hausdorff space is properly discontinuous.* ■

If G is the group of covering transformations of a covering projection, then a simple verification shows that G is properly discontinuous. We now show that any properly discontinuous group of homeomorphisms defines a covering projection.

7 THEOREM *Let G be a properly discontinuous group of homeomorphisms of a space Y. Then the projection of Y to the orbit space Y/G is a covering projection. If Y is connected, this covering projection is regular and G is its group of covering transformations.*

PROOF Let $p: Y \to Y/G$ be the projection. Then p is continuous. It is an open map, for if U is an open set in Y, then $p^{-1}(p(U)) = \cup \{gU \mid g \in G\}$ is open in Y, and therefore pU is open in Y/G. Let U be an open subset of Y such that whenever gU meets $g'U$, then $g = g'$. We show that $p(U)$ is evenly covered by p. The hypothesis on U ensures that $\{gU \mid g \in G\}$ is a disjoint collection of open sets whose union is $p^{-1}(p(U))$. It suffices to prove that $p \mid gU$ is a bijection from gU to $p(U)$. If $y \in U$, then $p(gy) = p(y)$, so $p(gU) = p(U)$. If $p(gy_1) = p(gy_2)$, with $y_1, y_2 \in U$, there is $g' \in G$ such that $gy_1 = g'gy_2$.

Therefore gU meets $g'gU$, and $g = g'g$. Hence $g' = 1_Y$ and $gy_1 = gy_2$. We have proved that p is a homeomorphism of gU onto $p(U)$. Since G is properly discontinuous, the sets $p(U)$ evenly covered by p constitute an open covering of Y/G.

Because $p(gy) = p(y)$, we see that G is contained in the group of covering transformations of p. Since G is transitive on the fibers of p, it follows from theorem 2.2.2 that if Y is connected, G equals the group of covering transformations. Since the group of covering transformations is transitive on each fiber, the covering projection is regular. ■

8 COROLLARY *Let G be a properly discontinuous group of homeomorphisms of a simply connected space Y. Then the fundamental group of the orbit space Y/G is isomorphic to G.*

PROOF By theorem 7, G is the group of covering transformations of the regular covering projection $p: Y \to Y/G$. By theorem 2, ψ is a monomorphism of G into the fundamental group of Y/G. Because G is transitive on the fibers of p, ψ is an isomorphism. ■

9 EXAMPLE Let $S^3 = \{(z_0,z_1) \in \mathbf{C}^2 \mid |z_0|^2 + |z_1|^2 = 1\}$ and let p and q be relatively prime integers. Define $h: S^3 \to S^3$ by

$$h(z_0,z_1) = (e^{2\pi i/p}z_0, e^{2\pi qi/p}z_1)$$

Then h is a homeomorphism of S^3 with period p (that is, $h^p = 1$), and $\mathbf{Z}_p$ acts on S^3 by

$$n(z_0,z_1) = h^n(z_0,z_1)$$

where n denotes the residue class of the integer n modulo p. In this way $\mathbf{Z}_p$ acts without fixed points on S^3. The orbit space of this action of $\mathbf{Z}_p$ on S^3 is called a *lens space* and is denoted by $L(p,q)$. By statement 6 and corollary 8, the fundamental group of $L(p,q)$ is isomorphic to $\mathbf{Z}_p$.

10 EXAMPLE Let $S^{2n+1} = \{(z_0,z_1, \ldots ,z_n) \in \mathbf{C}^{n+1} \mid \Sigma |z_i|^2 = 1\}$ and let $q_1, \ldots ,q_n$ be integers relatively prime to p. Define $h: S^{2n+1} \to S^{2n+1}$ by

$$h(z_0,z_1, \ldots ,z_n) = (e^{2\pi i/p}z_0, e^{2\pi i q_1/p}z_1, \ldots ,e^{2\pi i q_n/p}z_n)$$

Then, as in example 9, h determines an action of $\mathbf{Z}_p$ on S^{2n+1} without fixed points; the orbit space is called a *generalized lens space* and is denoted by $L(p,q_1, \ldots ,q_n)$. Its fundamental group is isomorphic to $\mathbf{Z}_p$.

It is possible to use theorem 7 to show that the projection $Y \to Y/G$ is a regular fibration with unique path lifting even when it may not be a covering projection. Note that if G acts on Y without fixed points, so does any subgroup of G, and if G' is a normal subgroup of Y, then G/G' acts without fixed points on Y/G'.

11 THEOREM *Let G be a group of homeomorphisms acting without fixed points on a path-connected space Y and assume that there is a decreasing sequence of subgroups*

$$G = G_0 \supset G_1 \supset \cdots \supset G_n \supset G_{n+1} \supset \cdots$$

such that

(*a*) $\cap G_n = \{1_Y\}$
(*b*) G_{n+1} *is a normal subgroup of* G_n *for* $n \geq 0$
(*c*) G_n/G_{n+1} *is a properly discontinuous group of homeomorphisms on* Y/G_{n+1} *and the projection* $Y \to Y/G_n$ *is a closed map for* $n \geq 0$
(*d*) *Any orbit of* Y *under* G_n *for* $n \geq 0$ *is compact*

Then the projection $p: Y \to Y/G$ *is a regular fibration with unique path lifting whose group of self-equivalences is* G.

PROOF Since $Y/G_n = (Y/G_{n+1})/(G_n/G_{n+1})$, it follows from (*c*) and theorem 7 that the projection

$$p_{n+1}: Y/G_{n+1} \to Y/G_n$$

is a regular covering projection for $n \geq 0$. Let

$$\tilde{Y} = \{(y_n) \in \times (Y/G_n) \mid p_{n+1}(y_{n+1}) = y_n \text{ for } n \geq 0\}$$

and define $\tilde{p}: \tilde{Y} \to Y/G$ by $\tilde{p}((y_n)) = y_0$. It is easy to verify that $\tilde{p}$ is a fibration with unique path lifting (it is the fibered product of the maps $\{p_1 \circ \cdots \circ p_j\}$).

For $n \geq 0$ there is a continuous closed projection map $\varphi_n: Y \to Y/G_n$ such that $p_{n+1} \circ \varphi_{n+1} = \varphi_n$. Therefore there is a continuous closed map $\varphi: Y \to \tilde{Y}$ defined by $\varphi(y) = (\varphi_n(y))$ and such that $\tilde{p} \circ \varphi = p$. To prove that φ is a homeomorphism, it suffices to show that it is a bijection. If $\varphi(y) = \varphi(y')$, then for $n \geq 0$ there is $g_n \in G_n$ such that $y = g_n y'$. Then $g_n y' = g_m y'$ for all m and n, and because G acts without fixed points, $g_m = g_n$ for all m and n. Therefore $g_n \in G_m$ for all m, and by (*a*), $g_n = 1_Y$. It follows that $y = y'$, and hence that φ is injective.

If $(y_n) \in \tilde{Y}$, then $\varphi_n{}^{-1} y_n$ is an orbit of Y under G_n. By (*d*), $\varphi_n{}^{-1} y_n$ is compact. Since

$$\varphi_n{}^{-1} y_n = \varphi_{n+1}^{-1} p_{n+1}^{-1} y_n \supset \varphi_{n+1}^{-1} y_{n+1}$$

the collection $\{\varphi_n{}^{-1} y_n\}$ consists of compact sets having the finite-intersection property. Therefore $\cap \varphi_n{}^{-1} y_n \neq \varnothing$. If $y \in \cap \varphi_n{}^{-1} y_n$, then $\varphi(y) = (y_n)$, showing that φ is surjective.

We have shown that $\varphi: Y \to \tilde{Y}$ is a homeomorphism. Therefore $p: Y \to Y/G$ is a fibration with unique path lifting. Since each element of G is a self-equivalence of p, the group of self-equivalences of p is transitive on each fiber. By corollary 3, p is a regular fibration and G is the group of self-equivalences of p. ∎

7 FIBER BUNDLES

A covering space is locally the product of its base space and a discrete space. This is generalized by the concept of fiber bundle, defined in this section, because the total space of a fiber bundle is locally the product of its base space and its fiber. The main result is that the bundle projection of a fiber bundle is a fibration.[1]

A *fiber bundle* $\xi = (E,B,F,p)$ consists of a *total space* E, a *base space* B, a *fiber* F, and a *bundle projection* $p\colon E \to B$ such that there exists an open covering $\{U\}$ of B and, for each $U \in \{U\}$, a homeomorphism $\varphi_U\colon U \times F \to p^{-1}(U)$ such that the composite

$$U \times F \xrightarrow{\varphi_U} p^{-1}(U) \xrightarrow{p} U$$

is the projection to the first factor. Thus the bundle projection $p\colon E \to B$ and the projection $B \times F \to B$ are locally equivalent. The *fiber over* $b \in B$ is defined to equal $p^{-1}(b)$, and we note that F is homeomorphic to $p^{-1}(b)$ for every $b \in B$. Usually there is also given a structure group G for the bundle consisting of homeomorphisms of F, and we define this concept next.

Let G be a group of homeomorphisms of F. Given a space F' and a collection $\Phi = \{\varphi\}$ of homeomorphisms $\varphi\colon F \to F'$, define $\varphi g\colon F \to F'$ for $\varphi \in \Phi$ and $g \in G$ by $\varphi g(y) = \varphi(gy)$ for $y \in F$. The collection Φ is called a *G structure* on F' if

(*a*) Given $\varphi \in \Phi$ and $g \in G$, then $\varphi g \in \Phi$
(*b*) Given $\varphi_1, \varphi_2 \in \Phi$, there is $g \in G$ such that $\varphi_1 = \varphi_2 g$

Condition (*a*) implies that G acts on the right on Φ, and condition (*b*) implies that this action of G is transitive on Φ. A fiber bundle (E,B,F,p) is said to have *structure group* G if each fiber $p^{-1}(b)$ has a G structure $\Phi(b)$ such that there exists an open covering $\{U\}$ of B and, for each $U \in \{U\}$, a homeomorphism $\varphi_U\colon U \times F \to p^{-1}(U)$ such that for $b \in U$, the map $F \to p^{-1}(b)$ sending x to $\varphi_U(b,x)$ is in $\Phi(b)$. It is clear that a given fiber bundle can always be given the structure of a fiber bundle with structure group the group of all homeomorphisms of F. It is also clear that a given fiber bundle can sometimes be given the structure of a fiber bundle with two different structure groups of homeomorphisms of F.

An *n-plane bundle,* or *real vector bundle,* is a fiber bundle whose fiber is $\mathbf{R}^n$ and whose structure group is the general linear group $GL(\mathbf{R}^n)$, which consists of all linear automorphisms of $\mathbf{R}^n$. A *complex n-plane bundle,* or *complex vector bundle,* is a fiber bundle whose fiber is $\mathbf{C}^n$ and whose structure group is $GL(\mathbf{C}^n)$.

We give some examples.

1 For spaces B and F the *product bundle* is the fiber bundle $(B \times F, B, F, p)$, where $p\colon B \times F \to B$ is projection to the first factor (it has the trivial group as structure group).

2 Given that $p\colon \tilde{X} \to X$ is a covering projection and X is a connected and locally path connected space, if $x_0 \in X$, then $(\tilde{X},X,p^{-1}(x_0),p)$ is a fiber bundle (and if X is path connected, it can be given the structure of a fiber bundle with

[1] For the general theory of fiber bundles see N. E. Steenrod, *The Topology of Fibre Bundles,* Princeton University Press, Princeton, N.J., 1951.

structure group $\pi(X,x_0)$, where $\pi(X,x_0)$ acts on $p^{-1}(x_0)$ by $[\omega]\tilde{x} = \tilde{x}[\omega]^{-1}$, with the right-hand side as in the proof of theorem 2.3.9).

3 Given that M is a differentiable n-manifold and $T(M)$ is the set of all tangent vectors to M, there is a fiber bundle $(T(M),M,\mathbf{R}^n,p)$, where $p\colon T(M) \to M$ assigns to each tangent vector its origin. This is called the *tangent bundle* and is denoted by $\tau(M)$. Because it can be given the structure group $GL(\mathbf{R}^n)$, it is an n-plane bundle, and if M is a complex manifold of complex dimension m, then $\tau(M)$ is a complex m-plane bundle.

4 Given that H is a closed subgroup of a Lie group G and that G/H is the quotient space of left cosets and $p\colon G \to G/H$ the projection, then $(G,G/H,H,p)$ is a fiber bundle (having structure group H acting on itself by left translation).

5 Represent S^n as the union of closed hemispheres E^n_- and E^n_+ with intersection S^{n-1} and let G be a group of homeomorphisms of a space F. Given a map $\varphi\colon S^{n-1} \to G$ such that the map $S^{n-1} \times F \to F$ sending (x,y) to $\varphi(x)y$ is continuous, let E_φ be the space obtained from $(E^n_- \times F) \vee (E^n_+ \times F)$ by identifying $(x,y) \in E^n_- \times F$ with $(x,\varphi(x)y) \in E^n_+ \times F$ for $x \in S^{n-1}$ and $y \in F$. These identifications are compatible with the projections $E^n_- \times F \to E^n_-$ and $E^n_+ \times F \to E^n_+$. Therefore there is a map $p_\varphi\colon E_\varphi \to S^n$ such that each of the composites

$$E^n_- \times F \to E_\varphi \xrightarrow{p_\varphi} S^n \qquad \text{and} \qquad E^n_+ \times F \to E_\varphi \xrightarrow{p_\varphi} S^n$$

is projection to the first factor. Then $(E_\varphi,S^n,F,p_\varphi)$ is a fiber bundle (having structure group G) which is said to be defined by the *characteristic map* φ.

6 Let $P_n(\mathbf{C})$ be the n-dimensional complex projective space coordinatized by homogeneous coordinates. If $z_0, z_1, \ldots, z_n \in \mathbf{C}$ are not all zero, let $[z_0,z_1, \ldots ,z_n] \in P_n(\mathbf{C})$ be that point of $P_n(\mathbf{C})$ having homogeneous coordinates $z_0, z_1, \ldots, z_n$. Regard S^{2n+1} as the set $\{(z_0,z_1, \ldots ,z_n) \in \mathbf{C}^{n+1} \mid \Sigma\, |z_i|^2 = 1\}$ and define $p\colon S^{2n+1} \to P_n(\mathbf{C})$ by $p(z_0,z_1, \ldots ,z_n) = [z_0,z_1, \ldots ,z_n]$. If $U_i \subset P_n(\mathbf{C})$ is the subset of points having a nonzero ith homogeneous coordinate, it is easy to see that $p^{-1}(U_i)$ is homeomorphic to $U_i \times S^1$. Therefore there is a fiber bundle $(S^{2n+1},P_n(\mathbf{C}),S^1,p)$ (having structure group S^1 acting on itself by left translation), and this is called the *Hopf bundle.*

7 If $\mathbf{Q}$ is the division ring of quaternions, there is a similar map $p\colon S^{4n+3} \to P_n(\mathbf{Q})$ and a *quaternionic Hopf bundle* $(S^{4n+3},P_n(\mathbf{Q}),S^3,p)$ (having structure group S^3 acting on itself by left translation).

The structure group will not be important for our purposes. Thus we define an *n-sphere bundle* to be a fiber bundle whose fiber is S^n [usually it is also required that it have as structure group the *orthogonal group* $0(n + 1)$ of all isometries in $GL(\mathbf{R}^{n+1})$]. If ξ is an n-sphere bundle, we shall denote its total space by $\dot{E}_\xi$. The mapping cylinder of the bundle projection $\dot{E}_\xi \to B$ is the total space E_ξ of a fiber bundle (E_ξ,B,E^{n+1},p_ξ), where $p_\xi\colon E_\xi \to B$ is the retraction of the mapping cylinder to B (and $p_\xi \mid \dot{E}_\xi\colon \dot{E}_\xi \to B$ is the original bundle projection).

If $\xi = (E,B,\mathbf{R}^{n+1},p)$ is an $(n + 1)$-plane bundle having structure group $0(n + 1)$, it is possible to introduce a norm in each fiber $p^{-1}(b)$. The subset $E' \subset E$ of all elements in E having unit norm is the total space of an n-sphere bundle $(E', B, S^n, p \mid E')$ called the *unit n-sphere bundle* of ξ. If the base space B of an $(n + 1)$-plane bundle is a paracompact Hausdorff space, the bundle can always be given $0(n + 1)$ as structure group. In particular, there is a *unit tangent bundle* of a paracompact differentiable manifold.

Two fiber bundles (E_1,B,F,p_1) and (E_2,B,F,p_2) with the same fiber and same base are said to be *equivalent* if there is a homeomorphism h: $E_1 \to E_2$ such that $p_2 \circ h = p_1$. If they both have structure group G, they are *equivalent over* G if there is a homeomorphism h as above, with the additional property that if $\varphi \in \Phi_1(b)$, then $h \circ \varphi \in \Phi_2(b)$ for $b \in B$. A fiber bundle is said to be *trivial* if it is equivalent to the product bundle of example 1 (or, equivalently, if it can be given the trivial group as structure group).

In view of example 2, fiber bundles are related to covering spaces in much the same way that fibrations are related to fibrations with unique path lifting. The rest of this section is devoted to a proof of the fact that in a fiber bundle (E,B,F,p) whose base space B is a paracompact Hausdorff space the map p is a fibration.

A map p: $E \to B$ is called a *local fibration* if there is an open covering $\{U\}$ of B such that $p \mid p^{-1}(U)$: $p^{-1}(U) \to U$ is a fibration for every $U \in \{U\}$. It is clear that a fibration is a local fibration[1] and that any bundle projection is a local fibration.

Given a map p: $E \to B$, we define a subspace $\bar{B} \subset E \times B^I$ by

$$\bar{B} = \{(e,\omega) \in E \times B^I \mid \omega(0) = p(e)\}$$

There is a map $\bar{p}$: $E^I \to \bar{B}$ defined by $\bar{p}(\bar{\omega}) = (\bar{\omega}(0), p \circ \bar{\omega})$ for $\bar{\omega}$: $I \to E$. A *lifting function* for p is a map

$$\lambda: \bar{B} \to E^I$$

which is a right inverse of $\bar{p}$. Thus a lifting function assigns to each point $e \in E$ and path ω in B starting at $p(e)$ a path $\lambda(e,\omega)$ in E starting at e that is a lift of ω. The relation between lifting functions and fibrations is contained in the following theorem.

8 THEOREM *A map p: $E \to B$ is a fibration if and only if there exists a lifting function for p.*

PROOF The proof involves repeated use of theorem 2.8 in the Introduction. If p is a fibration, let f': $\bar{B} \to E$ and F: $\bar{B} \times I \to B$ be defined by $f'(e,\omega) = e$

[1] Our proof of the converse for paracompact Hausdorff spaces B can be found in W. Hurewicz, On the concept of fibre space, *Proceedings of the National Academy of Sciences, U.S.A.*, vol. 41, pp. 956–961 (1955). Another proof can be found in W. Huebsch, On the covering homotopy theorem, *Annals of Mathematics*, vol. 61, pp. 555–563 (1955). Generalizations and related questions are treated in A. Dold, Partitions of unity in the theory of fibrations, *Annals of Mathematics*, vol. 78, pp. 223–255 (1963).

and $F((e,\omega), t) = \omega(t)$. Then

$$F((e,\omega), 0) = \omega(0) = p(e) = (p \circ f')(e,\omega)$$

By the homotopy lifting property of p, there is a map $F'\colon \bar{B} \times I \to E$ such that $F'((e,\omega), 0) = f'(e,\omega) = e$ and $p \circ F' = F$. F' defines a lifting function λ for p by $\lambda(e,\omega)(t) = F'((e,\omega), t)$.

Conversely, if λ is a lifting function for p, let $f'\colon X \to E$ and $F\colon X \times I \to B$ be such that $F(x,0) = pf'(x)$. Let $g\colon X \to B^I$ be defined by $g(x)(t) = F(x,t)$. There is a map $F'\colon X \times I \to E$ such that $F'(x,t) = \lambda(f'(x),g(x))(t)$. Because $F'(x,0) = f'(x)$ and $p \circ F' = F$, p has the homotopy lifting property. ∎

Let $p\colon E \to B$ and let W be a subset of B^I. Let $\tilde{W}$ be defined by

$$\tilde{W} = \{(e,\omega,s) \in E \times W \times I \mid \omega(s) = p(e)\}$$

An *extended lifting function* over W is a map

$$\Lambda\colon \tilde{W} \to E^I$$

such that $p(\Lambda(e,\omega,s)(t)) = \omega(t)$ and $\Lambda(e,\omega,s)(s) = e$. Thus an extended lifting function is a function which lifts paths to paths that pass through a given point of E at a given parameter value. It is reasonable to expect the following relation between the existence of lifting functions and extended lifting functions.

9 LEMMA *A map $p\colon E \to B$ has a lifting function if and only if there is an extended lifting function over B^I.*

PROOF If Λ is an extended lifting function over B^I, a lifting function λ for p is defined by $\lambda(e,\omega) = \Lambda(e,\omega,0)$.

To prove the converse, given a path ω in B, let ω_s and ω^s be the paths in B defined by

$$\omega_s(t) = \begin{cases} \omega(s-t) & 0 \le t \le s \\ \omega(0) & s \le t \le 1 \end{cases}$$

$$\omega^s(t) = \begin{cases} \omega(s+t) & 0 \le t \le 1-s \\ \omega(1) & 1-s \le t \le 1 \end{cases}$$

The maps $(\omega,s) \to \omega_s$ and $(\omega,s) \to \omega^s$ are continuous maps $B^I \times I \to B^I$. Given a lifting function $\lambda\colon \bar{B} \to E^I$ for p, we define an extended lifting function Λ over B^I by

$$\Lambda(e,\omega,s)(t) = \begin{cases} \lambda(e,\omega_s)(s-t) & 0 \le t \le s \\ \lambda(e,\omega^s)(t-s) & s \le t \le 1 \end{cases} \quad ∎$$

The main step in proving that a local fibration is a fibration is the fitting together of extended lifting functions over various open subsets of B^I. For this we need an additional concept. A covering $\{W\}$ of a space X is said to be *numerable* if it is locally finite and if for each W there is a function $f_W\colon X \to [0,1]$ such that $W = \{x \in X \mid f_W(x) \neq 0\}$.

10 LEMMA *Let $p: E \to B$ be a map. If there is a numerable covering $\{W_j\}$ of B^I such that for each j there is an extended lifting function over W_j, then there is a lifting function for p.*

PROOF Let the indexing set be $J = \{j\}$ and for each j let $f_j: B^I \to I$ be a map such that $W_j = \{\omega \in B^I \mid f_j(\omega) \neq 0\}$. For any subset $\alpha \subset J$ let $W_\alpha = \bigcup_{j \in \alpha} W_j$ and define $f_\alpha: B^I \to \mathbf{R}$ by

$$f_\alpha(\omega) = \Sigma_{j \in \alpha} f_j(\omega)$$

(this is a finite sum and is continuous because $\{W_j\}$ is locally finite). Then $f_\alpha(\omega) \geq 0$ for $\omega \in B^I$ and

$$W_\alpha = \{\omega \in B^I \mid f_\alpha(\omega) \neq 0\}$$

We define $\bar{B}_\alpha = \{(e,\omega) \in \bar{B} \mid \omega \in W_\alpha\}$.

Consider the set of pairs (α,λ_α), where $\alpha \subset J$ and $\lambda_\alpha: \bar{B}_\alpha \to E^I$ is a lifting function over $\bar{B}_\alpha$ [that is, $\lambda_\alpha(e,\omega)(0) = e$ and $p\lambda_\alpha(e,\omega)(t) = \omega(t)$]. We define a partial order $\leq$ in this set by $(\alpha,\lambda_\alpha) \leq (\beta,\lambda_\beta)$ if $\alpha \subset \beta$ and $\lambda_\alpha(e,\omega) = \lambda_\beta(e,\omega)$ whenever $(e,\omega) \in \bar{B}_\alpha$ and $f_\alpha(\omega) = f_\beta(\omega)$ [so if $(e,\omega) \in \bar{B}_\alpha$ and $\lambda_\alpha(e,\omega) \neq \lambda_\beta(e,\omega)$, then $\omega \in W_j$ for some $j \in \beta - \alpha$].

To prove that every simply ordered subset $\{\alpha_i,\lambda_{\alpha i}\}$ has an upper bound, let $\beta = \bigcup \alpha_i$. We shall define $\lambda_\beta: \bar{B}_\beta \to E^I$ so that $(\alpha_i,\lambda_{\alpha i}) \leq (\beta,\lambda_\beta)$ for all i. Let U be any open subset of W_β meeting only finitely many W_j with $j \in \beta$, say $W_{j_1}, \ldots, W_{j_r}$ (W_β can be covered by such sets U). Choose i so that $j_1, \ldots, j_r$ all belong to α_i. Then if $\alpha_i \subset \alpha_k$, $f_{\alpha i} \mid U = f_{\alpha_k} \mid U$. Because $(\alpha_i,\lambda_{\alpha i}) \leq (\alpha_k,\lambda_{\alpha k})$, it follows that $\lambda_{\alpha_i}(e,\omega) = \lambda_{\alpha k}(e,\omega)$ for $(e,\omega) \in \bar{B}_{\alpha i}$, with $\omega \in U$. Therefore there exists a map $\lambda_\beta: \bar{B}_\beta \to E^I$ such that $\lambda_\beta(e,\omega) = \lambda_{\alpha_i}(e,\omega)$ for α_i sufficiently large. We now show that $(\alpha_i,\lambda_{\alpha i}) \leq (\beta,\lambda_\beta)$. If $(e,\omega) \in \bar{B}_{\alpha i}$ and $\lambda_{\alpha_i}(e,\omega) \neq \lambda_\beta(e,\omega)$, there exists α_k such that $(\alpha_i,\lambda_{\alpha i}) \leq (\alpha_k,\lambda_{\alpha k})$ and $\lambda_{\alpha_i}(e,\omega) \neq \lambda_{\alpha k}(e,\omega)$. This implies $\omega \in W_j$ for some $j \in \alpha_k - \alpha_i$. Therefore $\omega \in W_j$ for some $j \in \beta - \alpha_i$, hence $(\alpha_i,\lambda_{\alpha i}) \leq (\beta,\lambda_\beta)$.

By Zorn's lemma, there is a maximal element (α,λ_α). To complete the proof we need only show that $\alpha = J$. If $\alpha \neq J$, let $j_0 \in J - \alpha$ and let $\beta = \alpha \cup \{j_0\}$. Define $g: W_\beta \to \mathbf{R}$ by $g(\omega) = f_\alpha(\omega)/f_\beta(\omega)$. Then $0 \leq g(\omega) \leq 1$, $g(\omega) \neq 0 \Leftrightarrow \omega \in W_\alpha$, and $g(\omega) \neq 1 \Leftrightarrow \omega \in W_{j_0}$. Define $\mu: \bar{B}_{j_0} \to E$ by

$$\mu(e,\omega) = \begin{cases} e & 0 \leq g(\omega) \leq 1/3 \\ \lambda_\alpha(e,\omega)(2g(\omega) - 2/3) & 1/3 \leq g(\omega) \leq 2/3 \\ \lambda_\alpha(e,\omega)(g(\omega)) & 2/3 \leq g(\omega) \leq 1 \end{cases}$$

Then μ is continuous. Let Λ be an extended lifting function over W_{j_0} and define $\lambda_\beta: \bar{B}_\beta \to E^I$ by

$$\lambda_\beta(e,\omega)(t) = \begin{cases} \Lambda(e,\omega,0)(t) & & 0 \leq g(\omega) \leq 1/3 \\ \lambda_\alpha(e,\omega)(t) & 0 \leq t \leq 2g(\omega) - 2/3 & \multirow{2}{*}{$1/3 \leq g(\omega) \leq 2/3$} \\ \Lambda(\mu(e,\omega),\ \omega,\ 2g(\omega) - 2/3)(t) & 2g(\omega) - 2/3 \leq t \leq 1 & \\ \lambda_\alpha(e,\omega)(t) & 0 \leq t \leq g(\omega) & \multirow{2}{*}{$2/3 \leq g(\omega) \leq 1$} \\ \Lambda(\mu(e,\omega),\ \omega,\ g(\omega))(t) & g(\omega) \leq t \leq 1 & \end{cases}$$

Then λ_β is a well-defined lifting function over W_β. Moreover, for $(e,\omega) \in \bar{B}_\alpha$, if $\lambda_\alpha(e,\omega) \neq \lambda_\beta(e,\omega)$, then $g(\omega) \neq 1$ and $\omega \in W_{j_0}$. Since $j_0 \in \beta - \alpha$, this means that $(\alpha,\lambda_\alpha) < (\beta,\lambda_\beta)$, contradicting the maximality of (α,λ_α). ∎

In case p has unique path lifting, lemma 10 would hold for any open covering $\{W_j\}$ of B^I such that there is a lifting function over W_j for each j (because the uniqueness of lifted paths enables the extended liftings to be amalgamated to a lifting for p). This was used in the proof of the theorem that a covering projection is a fibration (theorem 2.2.3), which was valid without any assumption on the base space.

11 LEMMA *Given a map $p\colon E \to B$ and subsets $U_1, \ldots, U_k$ of B such that there is an extended lifting function over $U_1{}^I, U_2{}^I, \ldots, U_k{}^I$, let W be the subset of B^I defined by*

$$W = \left\{\omega \in B^I \mid \omega\left(\left[\frac{i-1}{k}, \frac{i}{k}\right]\right) \subset U_i \text{ for } i = 1, \ldots, k\right\}$$

Then there is an extended lifting function over W.

PROOF Let Λ_i be an extended lifting function over $U_i{}^I$ for $i = 1, \ldots, k$. Given a path $\omega \in W$, let ω_i be the path equal to ω on $[(i-1)/k, i/k]$ and constant on $[0, (i-1)/k]$ and on $[i/k, 1]$. Given $(e,\omega,s) \in \tilde{W}$ such that $(n-1)/k \leq s \leq n/k$, define $e_i \in E$ for $i = 0, \ldots, k$ inductively so that

$$e_{n-1} = \Lambda_n(e,\omega_n,s)\left(\frac{n-1}{k}\right)$$

$$e_n = \Lambda_n(e,\omega_n,s)\left(\frac{n}{k}\right)$$

and

$$e_{i-1} = \Lambda_i\left(e_i,\omega_i,\frac{i}{k}\right)\left(\frac{i-1}{k}\right) \qquad 0 < i < n-1$$

$$e_{i+1} = \Lambda_{i+1}\left(e_i,\omega_{i+1},\frac{i}{k}\right)\left(\frac{i+1}{k}\right) \qquad n < i+1 \leq k$$

An extended lifting function Λ over W is defined by

$$\Lambda(e,\omega,s)(t) = \begin{cases} \Lambda_i\left(e_i,\omega_i,\dfrac{i}{k}\right)(t) & \dfrac{i-1}{k} \leq t \leq \dfrac{i}{k} \leq \dfrac{n-1}{k} \\ \Lambda_n(e,\omega_n,s)(t) & \dfrac{n-1}{k} \leq t \leq \dfrac{n}{k} \\ \Lambda_{i+1}\left(e_i,\omega_{i+1},\dfrac{i}{k}\right)(t) & \dfrac{n}{k} \leq \dfrac{i}{k} \leq t \leq \dfrac{i+1}{k} \end{cases}$$

We are now ready for the main result on the passage from a local fibration to a fibration.

12 THEOREM *Given a map $p\colon E \to B$ and a numerable covering $\mathcal{U}$ of B such that for $U \in \mathcal{U}$, $p \mid p^{-1}(U)\colon p^{-1}(U) \to U$ is a fibration, then p is a fibration.*

PROOF Let $\mathcal{U} = \{U_j\}$ and for $k \geq 1$, given a set of indices $j_1, \ldots, j_k$, let $W_{j_1 j_2 \ldots j_k}$ be the subset of B^I defined by

$$W_{j_1 j_2 \ldots j_k} = \left\{\omega \in B^I \mid \omega\left(\left[\frac{i-1}{k}, \frac{i}{k}\right]\right) \subset U_{j_i}, i = 1, \ldots, k\right\}$$

It is then clear that the collection $\{W_{j_1 j_2 \ldots j_k}\}$ (with k varying) is an open covering of B^I, and by lemma 11, each set $W_{j_1 j_2 \ldots j_k}$ has an extended lifting function. For k fixed the collection $\{W_{j_1 j_2 \ldots j_k}\}$ is locally finite. In fact, if $\omega \in B^I$, for each $i = 1, \ldots, k$ there is a neighborhood V_i of $\omega([(i-1)/k, i/k])$ meeting only finitely many U_j. Then $\bigcap_{1 \le i \le k} \langle [(i-1)/k, i/k], V_i \rangle$ is a neighborhood of ω meeting only finitely many $\{W_{j_1 j_2 \ldots j_k}\}$.

For each j let $f_j\colon B \to I$ be a continuous map such that $f_j(b) \neq 0$ if and only if $b \in U_j$. Define $\bar{f}_{j_1 \ldots j_k}\colon B^I \to I$ by

$$\bar{f}_{j_1 \ldots j_k}(\omega) = \inf \left\{ f_{j_i}\omega(t) \;\middle|\; \frac{i-1}{k} \le t \le \frac{i}{k}, i = 1, \ldots, k \right\}$$

Then $\bar{f}_{j_1 \ldots j_k}(\omega) \neq 0$ if and only if $\omega \in W_{j_1 \ldots j_k}$.

Unfortunately, the collection $\{W_{j_1 j_2 \ldots j_k}\}$ (all k) is not locally finite, otherwise the proof would be complete by lemma 10. This difficulty is circumvented by modifying the sets $W_{j_1 j_2 \ldots j_k}$. Since for fixed m the collection $\{W_{j_1 \ldots j_k}\}$ with $k < m$ is locally finite, the sum of the functions $\bar{f}_{j_1 \ldots j_k}$ with $k < m$ is a continuous real-valued function g_m on B^I. Define

$$f'_{j_1 \ldots j_m} = \inf(\sup(0, \bar{f}_{j_1 \ldots j_m} - m g_m), 1)$$

Then $f'_{j_1 \ldots j_m}\colon B^I \to I$ and we define $W'_{j_1 \ldots j_m} = \{\omega \in B^I \mid f'_{j_1 \ldots j_m}(\omega) \neq 0\}$. Clearly, $W'_{j_1 \ldots j_m} \subset W_{j_1 \ldots j_m}$; therefore there is an extended lifting function over $W'_{j_1 \ldots j_m}$. To complete the proof, it follows from lemma 10 that we need only verify that $\{W'_{j_1 \ldots j_k}\}$ (with k varying) is a locally finite covering of B^I.

For $\omega \in B^I$, let m be the smallest integer such that $\bar{f}_{j_1 \ldots j_m}(\omega) \neq 0$ for some $j_1, \ldots, j_m$. Then $g_m(\omega) = 0$ and $f'_{j_1 \ldots j_m}(\omega) = \bar{f}_{j_1 \ldots j_m}(\omega) \neq 0$. Therefore $\omega \in W'_{j_1 \ldots j_m}$, proving that $\{W'_{j_1 \ldots j_m}\}$ is a covering of B^I. To show that it is locally finite, assume N chosen so that $N > m$ and $\bar{f}_{j_1 \ldots j_m}(\omega) > 1/N$. Then $g_N(\omega) > 1/N$ and $Ng_N(\omega) > 1$. Hence $Ng_N(\omega') > 1$ for all ω' in some neighborhood V of ω. Therefore all functions $f'_{j_1 \ldots j_k}$ with $k \ge N$ vanish on V. But this means that the corresponding set $W'_{j_1 \ldots j_k}$ is disjoint from V. Since the collection $\{W'_{j_1 \ldots j_k}\}$ with $k < N$ is locally finite, the collection $\{W'_{j_1 \ldots j_k}\}$ (all k) is locally finite. ∎

The fact that any open covering of a paracompact Hausdorff space has a numerable refinement, leads to our next theorem.

13 THEOREM *If B is a paracompact Hausdorff space, a map $p\colon E \to B$ is a fibration if and only if it is a local fibration.* ∎

A bundle projection is a local fibration. Therefore, we have the following corollary.

14 COROLLARY *If (E,B,F,p) is a fiber bundle with base space B paracompact and Hausdorff, then p is a fibration.* ∎

8 FIBRATIONS

This section contains a general discussion of fibrations. We establish a relation between cofibrations and fibrations which allows the construction of fibrations from cofibrations by means of function spaces. We also prove that every map is equivalent, up to homotopy, to a map that is a fibration (this dualizes a similar result concerning cofibrations). The section contains definitions of the concepts of fiber homotopy type and induced fibration and a proof of the result that homotopic maps induce fiber-homotopy-equivalent fibrations.

We begin with an analogue of theorem 2.7.8 for cofibrations. Given a map $f\colon X' \to X$, let $\bar{X}$ be the quotient space of the sum $(X' \times I) \vee (X \times 0)$, obtained by identifying $(x',0) \in X' \times I$ with $(f(x'),0) \in X \times 0$ for all $x' \in X'$. We use $[x',t]$ and $[x,0]$ to denote the points of $\bar{X}$ corresponding to $(x',t) \in X' \times I$ and $(x,0) \in X \times 0$, respectively. Then $[x',0] = [f(x'),0]$. There is a map

$$\bar{i}\colon \bar{X} \to X \times I$$

defined by

$$\begin{aligned} \bar{i}[x',t] &= (f(x'),t) \qquad && x' \in X',\ t \in I \\ \bar{i}[x,0] &= (x,0) && x \in X \end{aligned}$$

A *retracting function* for f is a map

$$\rho\colon X \times I \to \bar{X}$$

which is a left inverse of $\bar{i}$. In case f is a closed inclusion map, so is $\bar{i}$, and a retracting function for f is a retraction of $X \times I$ to the subspace $X' \times I \cup X \times 0$.

1 THEOREM *A map $f\colon X' \to X$ is a cofibration if and only if there exists a retracting function for f.*

PROOF If f is a cofibration, let $g\colon X \to \bar{X}$ and $G\colon X' \times I \to \bar{X}$ be the maps defined by $g(x) = [x,0]$ and $G(x',t) = [x',t]$. Because

$$G(x',0) = [x',0] = [f(x'),0] = gf'(x)$$

it follows from the fact that f is a cofibration that there exists a map $\rho\colon X \times I \to \bar{X}$ such that $\rho(x,0) = g(x)$ and $\rho(f(x'),t) = G(x',t)$. Then ρ is a retracting function for f.

Conversely, given maps $g\colon X \to Y$ and $G\colon X' \times I \to Y$ such that $G(x',0) = gf(x')$ for $x' \in X'$, define

$$\bar{G}\colon \bar{X} \to Y$$

by $\bar{G}[x',t] = G(x',t)$ and $\bar{G}[x,0] = g(x)$. If $\rho\colon X \times I \to \bar{X}$ is a retracting function for f, the map $F = \bar{G} \circ \rho\colon X \times I \to Y$ has the properties $F(x,0) = g(x)$ and $F(f(x'),t) = G(x',t)$, showing that f is a cofibration. ∎

This leads to the following construction of fibrations from cofibrations.

2 THEOREM *Let $f\colon X' \to X$ be a cofibration, where X' and X are locally compact Hausdorff spaces, and let Y be any space. Then the map $p\colon Y^X \to Y^{X'}$ defined by $p(g) = g \circ f$ is a fibration.*

PROOF Let $\rho\colon X \times I \to \bar{X}$ be a retracting function for f (which exists by theorem 1). Then ρ defines a map

$$\rho'\colon Y^{\bar{X}} \to Y^{X \times I}$$

such that $\rho'(g) = g \circ \rho$ for $g\colon \bar{X} \to Y$. Because X' and X are locally compact

Hausdorff spaces, so is $\bar{X}$, and by theorem 2.9 in the introduction, $Y^{X \times I} \approx (Y^X)^I$ and

$$Y^{\bar{X}} \approx \{(g,G) \in Y^X \times (Y^{X'})^I \mid g \circ f = G(0)\}$$

Therefore ρ' corresponds to a lifting function for p: $Y^X \to Y^{X'}$, and by theorem 2.7.8, p is a fibration. ∎

3 COROLLARY *For any space Y let p: $Y^I \to Y \times Y$ be the map $p(\omega) = (\omega(0),\omega(1))$ for ω: $I \to Y$. Then p is a fibration.*

PROOF Because $\dot{I} \times I \cup I \times 0$ is a retract of $I \times I$, the inclusion map $\dot{I} \subset I$ is a cofibration [equivalently, the pair $(I,\dot{I})$ has the homotopy extension property with respect to any space]. The result follows from theorem 2 and the observation that $Y^{\dot{I}}$ is homeomorphic to $Y \times Y$ under the map $g \to (g(0),g(1))$ for g: $\dot{I} \to Y$. ∎

Let f: $B' \to B$ and p: $E \to B$ be maps and let E' be the subset of $B' \times E$ defined by

$$E' = \{(b',e) \in B' \times E \mid f(b') = p(e)\}$$

E' is called the *fibered product* of B' and E (more precisely, the fibered product of f and p; *cf.* Sec. 2.2). Note that there are maps p': $E' \to B'$ and f'; $E' \to E$ defined by $p'(b',e) = b'$ and $f'(b',e) = e$. E' and the maps p' and f' are characterized as the product of f: $B' \to B$ and p: $E \to B$ in the category whose objects are continuous maps with range B and whose morphisms are commutative triangles

$$\begin{tikzcd} X_1 \arrow[rr, "h"] \arrow[dr, "g_1"'] & & X_2 \arrow[dl, "g_2"] \\ & B & \end{tikzcd}$$

The following properties are easily verified.

4 *If p is injective (or surjective), so is p'.* ∎

5 *If p: $B \times F \to B$ is the trivial fibration, then p': $E' \to B'$ is equivalent to the trivial fibration $B' \times F \to B'$.* ∎

6 *If p is a fibration (with unique path lifting), so is p'.* ∎

7 *If p is a fibration, f can be lifted to E if and only if p' has a section.* ∎

Note that since the fibered product is symmetric in B and E (or rather, in f and p), there is a similar set of statements where p and p' are replaced by f and f'.

If p: $E \to B$ is a fibration (or covering projection) and f: $B' \to B$ is a map, then, by property 6 (or property 5), p': $E' \to B'$ is a fibration (or covering projection) and is called the *fibration induced* from p by f (or *covering projection induced* from p by f). If $\xi = (E,B,F,p)$ is a fiber bundle and f: $B' \to B$ is a map, it follows from property 5 that there is a fiber bundle (E',B',F,p'). This is called the *fiber bundle induced* from ξ by f and is denoted by $f^*\xi$. In the case of an inclusion map i: $B' \subset B$ we use $E \mid B'$ to denote the fibered product of B' and E, and if ξ is a fiber bundle with base space B, $\xi \mid B'$ will denote the fiber bundle with base space B' induced by i. Observe that $\xi \mid B'$ is equivalent to $(p^{-1}(B'), B', F, p \mid p^{-1}(B'))$.

8 COROLLARY *For any space Y and point $y_0 \in Y$, let p: $P(Y,y_0) \to Y$ be the map sending each path starting at y_0 to its endpoint. Then p is a fibration whose fiber over y_0 is the loop space ΩY.*

PROOF Let f: $Y \to Y \times Y$ be defined by $f(y) = (y_0,y)$ and let $\bar{p}$: $Y^I \to Y \times Y$ be the fibration of corollary 3. The fibration induced by f is equivalent to the map p: $P(Y,y_0) \to Y$, where $p(\omega) = \omega(1)$, and $p^{-1}(y_0)$ the fiber over y_0, is by definition, the loop space ΩY. ■

It follows from corollary 3 that the map p': $Y^I \to Y$ defined by $p'(\omega) = \omega(0)$ [or by $p'(\omega) = \omega(1)$] is a fibration, because it is the composite of fibrations $Y^I \to Y \times Y \to Y$. If p: $E \to B$ is any map and p': $B^I \to B$ is the fibration defined by $p'(\omega) = \omega(0)$, then the fibered product of E and B^I is just the space $\bar{B}$ used to define the concept of lifting function for p.

These remarks about fibered products and induced fibrations have analogues for cofibrations. Given maps f_1: $X \to X_1$ and f_2: $X \to X_2$, the *cofibered sum* of X_1 and X_2 is the quotient space X' of $X_1 \vee X_2$ obtained by identifying $f_1(x)$ with $f_2(x)$ for all $x \in X$. There are maps i_1: $X_1 \to X'$ and i_2: $X_2 \to X'$, and these characterize X' as the sum of f_1 and f_2 in the category whose objects are maps with domain X and whose morphisms are commutative triangles. If f_1: $X \to X_1$ is a cofibration, so is i_2: $X_2 \to X'$, and this is called the *cofibration induced* from f_1 by f_2.

The map h_0: $X' \to X' \times I$ defined by $h_0(x') = (x',0)$ is a cofibration for any space X', and if f: $X' \to X$ is any map, the cofibered sum of $X' \times I$ and X is just the space $\bar{X}$ used to define the concept of retracting function for f.

Let p: $E \to B$ be a fibration. Maps f_0, f_1: $X \to E$ are said to be *fiber homotopic,* denoted by $f_0 \underset{p}{\simeq} f_1$, if there is a homotopy F: $f_0 \simeq f_1$ such that $pF(x,t) = pf_0(x)$ for $x \in X$ and $t \in I$ (in which case $p \circ f_0 = p \circ f_1$). This is an equivalence relation in the set of maps $X \to E$. The equivalence classes are denoted by $[X;E]_p$, and if f: $X \to E$, $[f]_p$ denotes its fiber homotopy class. The concept of fiber homotopy is dual to the concept of relative homotopy.

We use induced fibrations to prove that any map is, up to homotopy equivalence, a fibration. Let f: $X \to Y$ and let p': $Y^I \to Y$ be the fibration defined by $p'(\omega) = \omega(0)$. Let p: $P_f \to X$ be the fibration induced from p' by f. It is called the *mapping path fibration* of f and is dual to the mapping cylinder. There is a section s: $X \to P_f$ of p defined by $s(x) = (x,\omega_{f(x)})$, where $\omega_{f(x)}$ is the constant path in Y at $f(x)$. There is also a map p'': $P_f \to Y$ defined by $p''(x,\omega) = \omega(1)$. We then have the following dual of theorem 1.4.12.

9 THEOREM *Given a map f: $X \to Y$, there is a commutative diagram*

$$\begin{tikzcd} X \arrow[r, "s"] \arrow[dr, "f"'] & P_f \arrow[d, "p''"] \\ & Y \end{tikzcd}$$

such that

(*a*) $1_{P_f} \underset{p}{\simeq} s \circ p$

(*b*) p'' *is a fibration*

PROOF The triangle is commutative by the definition of the maps involved.

(*a*) Define $F: P_f \times I \to P_f$ by $F((x,\omega), t) = (x,\omega_{1-t})$, where $\omega_{1-t}(t') = \omega((1-t)t')$. Then F is a fiber homotopy from 1_{P_f} to $s \circ p$.

(*b*) Let $g: W \to P_f$ and $G: W \times I \to Y$ be such that $G(w,0) = p''g(w)$ for $w \in W$. Then there exist maps $g': W \to X$ and $g'': W \to Y^I$ such that $g''(w)(0) = fg'(w)$ and $g(w) = (g'(w),g''(w))$ for $w \in W$. We define a lifting $G': W \times I \to P_f$ of G beginning with g by $G'(w,t) = (g'(w), \bar{g}(w,t))$, where $\bar{g}(w,t) \in Y^I$ is defined by

$$\bar{g}(w,t)(t') = \begin{cases} g''(w)(2t'/(2-t)) & 0 \le 2t' \le 2-t \le 2,\ w \in W \\ G(w,2t'+t-2) & 1 \le 2-t \le 2t' \le 2,\ w \in W \end{cases}$$

Since p'' has the homotopy lifting property, it is a fibration. ∎

It follows that the fibration $p'': P_f \to Y$ is equivalent (by means of $s: X \to P_f$ and $p: P_f \to X$) in the homotopy category of maps with range Y to the original map $f: X \to Y$. In replacing f by an equivalent fibration, we replaced X by a space P_f of the same homotopy type, whereas in Sec. 1.4, when f was replaced by an equivalent cofibration, the space Y was replaced by a space Z_f of the same homotopy type.

Two fibrations $p_1: E_1 \to B$ and $p_2: E_2 \to B$ are said to be *fiber homotopy equivalent* (or to have the *same fiber homotopy type*) if there exist maps $f: E_1 \to E_2$ and $g: E_2 \to E_1$ preserving fibers in the sense that $p_2 \circ f = p_1$ and $p_1 \circ g = f_2$ and such that $g \circ f \underset{p_1}{\simeq} 1_{E_1}$ and $f \circ g \underset{p_2}{\simeq} 1_{E_2}$. Each of the maps f and g is called a *fiber homotopy equivalence*. The rest of this section is concerned with fiber homotopy equivalence.

We begin with the following result concerning liftings of homotopic maps.

10 THEOREM *Let $p: E \to B$ be a fibration and let $F_0, F_1: X \times I \to E$ be maps. Given homotopies $H: p \circ F_0 \simeq p \circ F_1$ and $G: F_0 \mid X \times 0 \simeq F_1 \mid X \times 0$ such that $H(x,0,t) = pG(x,0,t)$, there is a lifting $H': X \times I \times I \to E$ of H which is a homotopy from F_0 to F_1 and is an extension of G.*

PROOF Let $A = (I \times 0) \cup (0 \times I) \cup (I \times 1) \subset I \times I$ and define $f: X \times A \to E$ by

$$\begin{aligned} f(x,t,0) &= F_0(x,t) \\ f(x,0,t) &= G(x,0,t) \\ f(x,t,1) &= F_1(x,t) \end{aligned}$$

Then $H \mid X \times A = p \circ f$. Because there is a homeomorphism of $I \times I$ with itself taking A onto $I \times 0$, there is a homeomorphism of $X \times I \times I$ with itself taking $X \times A$ onto $X \times I \times 0$. It follows from the homotopy lifting property of p that there is a lifting $H': X \times I \times I \to E$ of H such that $H' \mid X \times A = f$. ∎

Taking H and G to be constant homotopies, we obtain the following corollary.

11 COROLLARY *Let p: $E \to B$ be a fibration and let F_0, F_1: $X \times I \to E$ be liftings of the same map such that $F_0 \mid X \times 0 = F_1 \mid X \times 0$. Then $F_0 \underset{p}{\simeq} F_1$* rel $X \times 0$. ■

Let p: $E \to B$ be a fibration and let ω: $I \to B$ be a path in its base space. By the homotopy lifting property of p, there exists a map F: $p^{-1}(\omega(0)) \times I \to E$ such that $pF(x,t) = \omega(t)$ and $F(x,0) = x$ for $x \in p^{-1}(\omega(0))$ and $t \in I$. Let f: $p^{-1}(\omega(0)) \to p^{-1}(\omega(1))$ be the map $f(x) = F(x,1)$. It follows from theorem 10 that if $\omega \simeq \omega'$ are homotopic paths in B and if F, F': $p^{-1}(\omega(0)) \times I \to E$ are such that $pF(x,t) = \omega(t)$, $pF'(x,t) = \omega'(t)$, and $F(x,0) = x = F'(x,0)$ for $x \in p^{-1}(\omega(0))$ and $t \in I$, then the maps f, f': $p^{-1}(\omega(0)) \to p^{-1}(\omega(1))$ defined by $f(x) = F(x,1)$ and $f'(x) = F'(x,1)$ are homotopic. Hence there is a well-defined homotopy class $[f] \in [p^{-1}(\omega(0));p^{-1}(\omega(1))]$ corresponding to a path class $[\omega]$ in B. We let $h[\omega] = [f]$.

The following is the form theorem 2.3.7 takes for an arbitrary fibration.

12 THEOREM *Let p: $E \to B$ be a fibration. There is a contravariant functor from the fundamental groupoid of B to the homotopy category which assigns to $b \in B$ the fiber over b and to a path class $[\omega]$ the homotopy class $h[\omega]$.*

PROOF If ω_b is the constant path at b, let F: $p^{-1}(b) \times I \to E$ be the map $F(x,t) = x$. The corresponding map f: $p^{-1}(b) \to p^{-1}(b)$ defined by $f(x) = F(x,1)$ is the identity map. Hence

$$h[\omega_b] = [1_{p^{-1}(b)}]$$

showing that h preserves identities.

Let ω and ω' be paths in B such that $\omega(1) = \omega'(0)$. Given a map F: $p^{-1}(\omega(0)) \times I \to E$ such that $F(x,0) = x$ and $pF(x,t) = \omega(t)$ for $x \in p^{-1}(\omega(0))$ and $t \in I$, and given F': $p^{-1}(\omega(1)) \times I \to E$ such that $F'(x',0) = x'$ and $pF'(x',t) = \omega'(t)$ for $x' \in p^{-1}(\omega'(0))$ and $t \in I$, let f: $p^{-1}(\omega(0)) \to p^{-1}(\omega'(0))$ be defined by $f(x) = F(x,1)$ and let F'': $p^{-1}(\omega(0)) \times I \to E$ be defined by

$$F''(x,t) = \begin{cases} F(x,2t) & 0 \le t \le ½, x \in p^{-1}(\omega(0)) \\ F'(f(x), 2t - 1) & ½ \le t \le 1, x \in p^{-1}(\omega(0)) \end{cases}$$

Then $pF''(x,t) = (\omega * \omega')(t)$ and $F''(x,0) = x$ for $x \in p^{-1}(\omega(0))$ and $t \in I$. Let f': $p^{-1}(\omega'(0)) \to p^{-1}(\omega'(1))$ be defined by $f'(x') = F'(x',1)$. Then $F''(x,1) = f'(f(x))$ for $x \in p^{-1}(\omega(0))$, which shows that

$$h[\omega * \omega'] = h[\omega'] * h[\omega]$$

Therefore h is a contravariant functor. ■

This yields the following analogue of corollary 2.3.8 for an arbitrary fibration.

13 COROLLARY *If p: $E \to B$ is a fibration with a path-connected base space, any two fibers have the same homotopy type.* ■

The following result asserts that homotopic maps induce fiber-homotopy-equivalent fibrations.

14 THEOREM *Let p: $E \to B$ be a fibration and let f_0, f_1: $X \to B$ be homotopic. The fibrations induced from p by f_0 and by f_1 are fiber homotopy equivalent.*

PROOF Let p_0: $E_0 \to X$ and p_1: $E_1 \to X$ be the fibrations induced from p by f_0 and f_1, respectively, and let f'_0: $E_0 \to E$ and f'_1: $E_1 \to E$ be the corresponding maps such that $p \circ f'_0 = f_0 \circ p_0$ and $p \circ f'_1 = f_1 \circ p_1$. Given a homotopy F: $X \times I \to B$ from f_0 to f_1, there are maps F'_0: $E_0 \times I \to E$ and F'_1: $E_1 \times I \to E$ such that $p \circ F'_0 = F \circ (p_0 \times 1_I)$ and $p \circ F'_1 = F \circ (p_1 \times 1_I)$ and $F'_0 \mid E_0 \times 0 = f'_0$ and $F'_1 \mid E_1 \times 1 = f'_1$. Let g_0: $E_0 \to E_1$ and g_1: $E_1 \to E_0$ be the fiber preserving maps defined by the property $F'_0(x,1) = f'_1 g_0(x)$ for $x \in E_0$ and $F'_1(y,0) = f'_0 g_1(y)$ for $y \in E_1$. Then

$$p \circ F'_0 \circ (g_1 \times 1_I) = F \circ (p_0 \times 1_I) \circ (g_1 \times 1_I) = F \circ (p_1 \times 1_I)$$

and
$$F'_0 \circ (g_1 \times 1_I) \mid E_1 \times 0 = F'_1 \mid E_1 \times 0$$

It follows from theorem 10 that $F'_1 \underset{p}{\simeq} F'_0 \circ (g_1 \times 1_I)$. In a similar fashion $F'_0 \underset{p}{\simeq} F'_1 \circ (g_0 \times 1_I)$. This implies that $g_0 g_1 \underset{p_1}{\simeq} 1_{E_1}$ and $g_1 g_0 \underset{p_0}{\simeq} 1_{E_0}$. ■

Clearly, a constant map induces a trivial fibration, and we have the following result.

15 COROLLARY *If p: $E \to B$ is a fibration and B is contractible, then p is fiber homotopy equivalent to the trivial fibration $B \times p^{-1}(b_0) \to B$ for any $b_0 \in B$.* ■

Let B be a space which is the join of some space Y with S^0. Then $B = C_-Y \cup C_+Y$, where C_-Y and C_+Y are cones over Y and $C_-Y \cap C_+Y = Y$. Let $y_0 \in Y$ and let p: $E \to B$ be a fibration with fiber $F_0 = p^{-1}(y_0)$. It follows from corollary 15 that there are fiber homotopy equivalences f_-: $C_-Y \times F_0 \to p^{-1}(C_-Y)$ and g_+: $p^{-1}(C_+Y) \to C_+Y \times F_0$. A *clutching function* μ: $Y \times F_0 \to F_0$ for p is a function μ defined by the equation

$$g_+ f_-(y,z) = (y, \mu(y,z)) \qquad y \in Y, z \in F_0$$

where f_-: $C_-Y \times F_0 \to p^{-1}(C_-Y)$ and g_+: $p^{-1}(C_+Y) \to C_+Y \times F_0$ are fiber homotopy equivalences. If C_-Y and C_+Y are contractible to y_0 relative to y_0, it follows from theorem 10 that f_- and g_+ can be chosen so that $z \to f_-(y_0,z)$ is homotopic to the map $F_0 \subset p^{-1}(C_-Y)$ and $z \to g_+(z)$ is homotopic to the map $z \to (y_0,z)$ of F_0 to $C_+Y \times F_0$. In this case the clutching function μ corresponding to f_- and g_+ has the property that the map $z \to \mu(y_0,z)$ is homotopic to the identity map $F_0 \subset F_0$.

Let E_φ be the fiber bundle over S^n defined by a characteristic map φ: $S^{n-1} \to G$, as in example 2.7.5 (where G is a group of homeomorphisms of the fiber F). Then $E^n_- = C_-S^{n-1}$ and $E^n_+ = C_+S^{n-1}$, and it is easy to verify that f_- and g_+ can be chosen so that the corresponding clutching function μ: $S^{n-1} \times F \to F$ is the map $\mu(x,z) = \varphi(x)z$.

EXERCISES

A LOCAL CONNECTEDNESS

1 Prove that a space X is locally path connected if and only if for any neighborhood U of x in X there exists a neighborhood V of x such that every pair of points in V can be joined by a path in U.

2 If X is a space, let $\bar{X}$ denote the set X retopologized by the topology generated by path components of open sets of X. Prove that $\bar{X}$ is locally path connected and that the identity map of X is a continuous function $j\colon \bar{X} \to X$ having the property that for any locally path-connected space Y a function $f\colon Y \to \bar{X}$ is continuous if and only if $j \circ f\colon Y \to X$ is continuous.

3 For any space X let $\bar{X}$ and $j\colon \bar{X} \to X$ be as in exercise 2. Prove that $j_\#\colon \pi(\bar{X},x_0) \approx \pi(X,x_0)$.

B COVERING SPACES

1 Let X be the union of two closed simply connected and locally path-connected subsets A and B such that $A \cap B$ consists of a single point. Prove that if $p\colon \tilde{X} \to X$ is a nonempty path-connected fibration with unique path lifting, then p is a homeomorphism.

2 Let $\tilde{X} = \{(x,y) \in \mathbf{R}^2 \mid x \text{ or } y \text{ an integer}\}$ and let

$$X = S^1 \vee S^1 = \{(z_1,z_2) \in S^1 \times S^1 \mid z_1 = 1 \text{ or } z_2 = 1\}$$

Prove that the map $p\colon \tilde{X} \to X$ such that $p(x,y) = (e^{2\pi i x},e^{2\pi i y})$ is a covering projection.

3 With $p\colon \tilde{X} \to X$ as in exercise 2 above, let $Y \subset \tilde{X}$ be defined by

$$Y = \{(x,y) \in \tilde{X} \mid 0 \leq x \leq 1, 0 \leq y \leq 1\}$$

Prove that Y is a retract of $\tilde{X}$ and that $(p \mid Y)_\#$ maps a generator of $\pi(Y)$ to the commutator of the two elements of $\pi(X)$ corresponding to the two circles of X.

4 Prove that $\pi(S^1 \vee S^1)$ is nonabelian.

C THE COVERING SPACE $ex\colon \mathbf{R} \to S^1$

1 For an arbitrary space X prove that a map $f\colon X \to S^1$ can be lifted to a map $\tilde{f}\colon X \to \mathbf{R}$ such that $f = ex \circ \tilde{f}$ if and only if f is null homotopic.

2 Let X be a connected locally path-connected space with base point $x_0 \in X$. Prove that the map

$$[X,x_0; S^1,1] \to \operatorname{Hom}(\pi(X,x_0), \pi(S^1,1))$$

which assigns to $[f]$ the homomorphism

$$f_\#\colon \pi(X,x_0) \to \pi(S^1,1)$$

is a monomorphism (the set of homotopy classes being a group by virtue of the group structure on S^1).

3 Prove that any two maps from a simply connected locally path-connected space to S^1 are homotopic.

4 Prove that any map of the real projective space P^n for $n \geq 2$ to S^1 is null homotopic.

5 Prove that there is no map $f\colon S^n \to S^1$ for $n \geq 2$ such that $f(-x) = -f(x)$.

6 *Borsuk-Ulam theorem.* Prove that if $f\colon S^2 \to \mathbf{R}^2$ is a map such that $f(-x) = -f(x)$, then there exists a point $x_0 \in S^2$ such that $f(x_0) = 0$.

D COVERING SPACES OF TOPOLOGICAL GROUPS

1 Let H be a subgroup of a topological group and let G/H be the homogeneous space of right cosets. Prove that the projection $G \to G/H$ is a covering projection if and only if H is discrete.

2 Prove that a connected locally path-connected covering space of a topological group can be given a group structure that makes it a topological group and makes the projection map a homomorphism.

A *local homomorphism* φ from one topological group G to another G' is a continuous map from some neighborhood U of e in G to G' such that if g_1, g_2, $g_1g_2 \in U$, then $\varphi(g_1g_2) = \varphi(g_1)\varphi(g_2)$. A *local isomorphism* from G to G' is a homeomorphism φ from some neighborhood U of e to some neighborhood U' of e' such that φ and φ^{-1} are both local homomorphisms (in which case G and G' are said to be *locally isomorphic*).

3 Prove that a continuous homomorphism $\varphi\colon G \to G'$ between connected topological groups is a covering projection if and only if there exists a neighborhood U of e in G such that $\varphi \mid U$ is a local isomorphism from G to G'.

4 Let φ be a local homomorphism from a connected topological group G to a topological group G' defined on a connected neighborhood U of e in G. Let $\tilde{G}$ be the subgroup of $G \times G'$ generated by the graph of φ (that is, generated by $\{(g,g') \in G \times G' \mid g' = \varphi(g), g \in U\}$). $\tilde{G}$ is topologized by taking as a base for neighborhoods of (e,e') the graph of $\varphi \mid N$ as N varies over neighborhoods of e in U. Prove that $\tilde{G}$ is a connected topological group, the projection $p_1\colon \tilde{G} \to G$ is a covering projection, and the projection $p_2\colon \tilde{G} \to G'$ is continuous.

5 Prove that two connected locally path-connected topological groups are locally isomorphic if and only if there is a topological group which is a covering space of each of them.

6 If G is a simply connected locally path-connected topological group and φ is a local homomorphism from G to a topological group G', prove that there is a continuous homomorphism $\varphi'\colon G \to G'$ which agrees with φ on some neighborhood of e in G.

E FIBRATIONS

1 If $p\colon E \to B$ is a fibration, prove that $p(E)$ is a union of path components of B.

2 If a fibration has path-connected base and some fiber is path connected, prove that its total space is also path connected.

3 Let $p\colon E \to B$ be a fibration and let X be a locally compact Hausdorff space. Define $p'\colon E^X \to B^X$ by $p'(g) = p \circ g$ for $g\colon X \to E$. Prove that p' is a fibration.

4 Let $p\colon E \to B$ be a fibration and let $b_0 \in p(E)$, $F = p^{-1}(b_0)$. Let X be a space regarded as a subset of some cone CX. Prove that the map

$$p_\#\colon [CX,X;\, E,F] \to [CX,X;\, B,b_0]$$

is a bijection.

5 Let $p\colon E \to B$ be a fibration and let $e_0 \in E$, $b_0 = p(e_0)$, and $F = p^{-1}(b_0)$. If B is simply connected, prove that $\pi(F,e_0) \to \pi(E,e_0)$ is an epimorphism.

6 Let $p: E \to B$ be a fibration and let $e_0 \in E$ and $b_0 = p(e_0)$. If $p^{-1}(b_0)$ is simply connected, prove that

$$p_\#: \pi(E,e_0) \approx \pi(B,b_0)$$

7 Let $p: E \to B$ be a fibration and $b_0 \in p(E)$. If E is simply connected, prove that there is a bijection between $\pi(B,b_0)$ and the set of path components of $p^{-1}(b_0)$.

CHAPTER THREE
POLYHEDRA

IN CHAPTER TWO THE FUNDAMENTAL GROUP FUNCTOR WAS USED TO CLASSIFY covering spaces. We now consider the problem of computing the fundamental group of a specific space. We shall show that the fundamental groups of many spaces (the class of polyhedra) can be described by means of generators and relations.

A polyhedron is a topological space which admits a triangulation by a simplicial complex. Thus we start with a study of the category of simplicial complexes. A simplicial complex consists of an abstract scheme of vertices and simplexes (each simplex being a finite set of vertices). Associated to such a simplicial complex is a topological space built by piecing together convex cells with identifications prescribed by the abstract scheme. Since the topological properties of these spaces are determined by the abstract scheme, the study of simplicial complexes and polyhedra is often called combinatorial topology.

A compact polyhedron admits a triangulation by a finite simplicial complex. Thus these spaces are effectively described in finite terms and serve as a useful class of spaces for questions involving computability of functors.

Sections 3.1 and 3.2 are devoted to definitions and elementary topological properties of polyhedra. Section 3.3 introduces the concept of subdivision of a simplicial complex, and it is shown that a compact polyhedron admits arbitrarily fine triangulations. This result is used in Sec. 3.4 to prove the simplicial-approximation theorem, which asserts that continuous maps from compact polyhedra to arbitrary polyhedra can be approximated by simplicial maps.

The technique of simplicial approximation is used in Sec. 3.5 to prove that the set of homotopy classes of continuous maps from a compact polyhedron to an arbitrary polyhedron can be described abstractly in terms of triangulations of the polyhedra. In Sec. 3.6 this result provides an abstract description of the fundamental group of a polyhedron as the edge-path group of a triangulation, which is used in Sec. 3.7 to obtain a system of generators and relations for the fundamental group of a polyhedron. It is also shown in Sec. 3.7 that the fundamental group functor provides a faithful representation of the homotopy category of connected one-dimensional polyhedra. Section 3.8 consists of applications of the results on the fundamental group, some examples of polyhedra, and a description of the fundamental group of an arbitrary surface.

1 SIMPLICIAL COMPLEXES

This section contains definitions of the category of simplicial complexes and of covariant functors from this category to the category of topological spaces.

A *simplicial complex* K consists of a set $\{v\}$ of *vertices* and a set $\{s\}$ of finite nonempty subsets of $\{v\}$ called *simplexes* such that

(*a*) Any set consisting of exactly one vertex is a simplex.
(*b*) Any nonempty subset of a simplex is a simplex.

A simplex s containing exactly $q + 1$ vertices is called a *q-simplex*. We also say that the *dimension* of s is q and write $\dim s = q$. If $s' \subset s$, then s' is called a *face* of s (a *proper face* if $s' \neq s$), and if s' is a p-simplex, it is called a *p-face* of s. If s is a q-simplex, then s is the only q-face of s, and a face s' of s is a proper face if and only if $\dim s' < q$. It is clear that any simplex has only a finite number of faces. Because any face of a face of s is itself a face of s, the simplexes of K are partially ordered by the face relation (written $s' \leq s$ if s' is a face of s).

It follows from condition (*a*) that the 0-simplexes of K correspond bijectively to the vertices of K. It follows from condition (*b*) that any simplex is determined by its 0-faces. Therefore K can be regarded as equal to the set of its simplexes, and we shall identify a vertex of K with the 0-simplex corresponding to it.

We list some examples.

1 The empty set of simplexes is a simplicial complex denoted by $\varnothing$.

2 For any set A the set of all finite nonempty subsets of A is a simplicial complex.

3 If s is a simplex of a simplicial complex K, the set of all faces of s is a simplicial complex denoted by $\bar{s}$.

4 If s is a simplex of a simplicial complex K, the set of all proper faces of s is a simplicial complex denoted by $\dot{s}$.

5 If K is a simplicial complex, its *q-dimensional skeleton* K^q is defined to be the simplicial complex consisting of all p-simplexes of K for $p \leq q$.

6 Given a set X and a collection $\mathcal{W} = \{W\}$ of subsets of X, the *nerve* of $\mathcal{W}$, denoted by $K(\mathcal{W})$, is the simplicial complex whose simplexes are finite nonempty subsets of $\mathcal{W}$ with nonempty intersection. Thus the vertices of $K(\mathcal{W})$ are the nonempty elements of $\mathcal{W}$.

7 If K_1 and K_2 are simplicial complexes, their *join* $K_1 * K_2$ is the simplicial complex defined by

$$K_1 * K_2 = K_1 \vee K_2 \cup \{s_1 \vee s_2 \mid s_1 \in K_1, s_2 \in K_2\}$$

Thus the set of vertices of $K_1 * K_2$ is the set sum of the set of vertices of K_1 and the set of vertices of K_2.

8 There is a simplicial complex whose set of vertices is $\mathbf{Z}$ and whose set of simplexes is

$$\{\{n\} \mid n \in \mathbf{Z}\} \cup \{\{n, n+1\} \mid n \in \mathbf{Z}\}$$

9 For $n \geq 1$ regard $\mathbf{Z}^n$ as partially ordered by the ordering of its coordinates (that is, given $x, x' \in \mathbf{Z}^n$, then $x \leq x'$ if for the ith coordinates $x_i \leq x_i$ in $\mathbf{Z}$). There is a simplicial complex whose set of vertices is $\mathbf{Z}^n$ and whose simplexes are finite nonempty totally ordered subsets $\{x^0, \ldots, x^q\}$ of $\mathbf{Z}^n$ (that is, $x^0 \leq x^1 \leq \cdots \leq x^q$) such that for all $1 \leq i \leq n$, $x_i^q - x_i^0 = 0$ or 1.

If K is a simplicial complex, its *dimension*, denoted by $\dim K$, is defined to equal -1 if K is empty, to equal n if K contains an n-simplex but no $(n+1)$-simplex, and to equal ∞ if K contains n-simplexes for all $n \geq 0$. Thus $\dim K = \sup \{\dim s \mid s \in K\}$. K is said to be *finite* if it contains only a finite number of simplexes. If K is finite, then $\dim K < \infty$; however, if $\dim K < \infty$, K need not be finite (example 8 is an infinite simplicial complex whose dimension is 1).

A *simplicial map* $\varphi\colon K_1 \to K_2$ is a function φ from the vertices of K_1 to the vertices of K_2 such that for any simplex $s \in K_1$ its image $\varphi(s)$ is a simplex of K_2. For any K there is an identity simplicial map $1_K\colon K \to K$ corresponding to the identity vertex map. Given simplicial maps $K_1 \xrightarrow{\varphi} K_2 \xrightarrow{\psi} K_3$, the composite simplicial map $\psi \circ \varphi\colon K_1 \to K_3$ corresponds to the composite vertex map. Therefore there is a category of simplicial complexes and simplicial maps.

A *subcomplex* L of a simplicial complex K, denoted by $L \subset K$, is a subset of K (that is, $s \in L \Rightarrow s \in K$) that is a simplicial complex. It is clear that a subset L of K is a subcomplex if and only if any simplex in K that is a face of a simplex of L is a simplex of L. If $L \subset K$, there is a simplicial inclusion map $i\colon L \subset K$.

A subcomplex $L \subset K$ is said to be *full* if each simplex of K having all its vertices in L itself belongs to L. There is a subcomplex N of K consisting of all simplexes of K with no vertex in L. Clearly, N is the largest subcomplex of K disjoint from L. If $s = \{v_0, v_1, \ldots, v_q\}$ is any simplex of K, then either no vertex of s is in L (in which case $s \in N$), or every vertex belongs to L (in which case, if L is full, $s \in L$), or the vertices can be enumerated so that $v_i \in L$ if $i \le p$ and $v_i \notin L$ if $i > p$, where $0 \le p < q$. In the latter case, $s = s' \cup s''$, where $s' = \{v_0, \ldots, v_p\}$ is in L, if L is full, and $s'' = \{v_{p+1}, \ldots, v_q\}$ is in N. Therefore we have the following result.

10 LEMMA *If L is a full subcomplex of K and N is the largest subcomplex of K disjoint from L, any simplex of K is either in N, or in L, or of the form $s' \cup s''$ for some $s' \in L$ and $s'' \in N$.* ■

There is a category of *simplicial pairs* (K,L) (that is, K is a simplicial complex and L is a subcomplex, possibly empty) and simplicial maps $\varphi\colon (K_1,L_1) \to (K_2,L_2)$ (that is, φ is a simplicial map $K_1 \to K_2$ such that $\varphi(L_1) \subset L_2$). The category of simplicial complexes is a full subcategory of the category of simplicial pairs. There is also a category of *pointed simplicial complexes* K (that is, K is a simplicial complex together with a distinguished *base vertex*) and simplicial maps preserving base vertices which is a full subcategory of the category of simplicial pairs. Following are some examples.

11 For any q the q-dimensional skeleton K^q is a subcomplex of K, and if $p \le q$, K^p is a subcomplex of K^q.

12 For any $s \in K$ there are subcomplexes $\dot{s} \subset \bar{s} \subset K$.

13 If $\{L_j\}_{j \in J}$ is a family of subcomplexes of K, then $\bigcap L_j$ and $\bigcup L_j$ are also subcomplexes of K.

14 Given that $A \subset X$, $\mathcal{W} = \{W\}$ is a collection of subsets of X, and $K_A(\mathcal{W})$ is the collection of finite nonempty subsets of $\mathcal{W}$ whose intersection meets A in a nonempty subset, then $K_A(\mathcal{W})$ is a subcomplex of the nerve $K(\mathcal{W})$.

We now define a covariant functor from the category of simplicial complexes and simplicial maps to the category of topological spaces and continuous maps. Given a nonempty simplicial complex K, let $|K|$ be the set of all functions α from the set of vertices of K to I such that

(*a*) For any α, $\{v \in K \mid \alpha(v) \ne 0\}$ is a simplex of K (in particular, $\alpha(v) \ne 0$

for only a finite set of vertices).

(*b*) For any α, $\Sigma_{v \in K} \alpha(v) = 1$.

If $K = \varnothing$, we define $|K| = \varnothing$.

The real number $\alpha(v)$ is called the vth *barycentric coordinate of* α. There is a metric d on $|K|$ defined by

$$d(\alpha,\beta) = \sqrt{\Sigma_{v \in K} [\alpha(v) - \beta(v)]^2}$$

and the topology on $|K|$ defined by this metric is called the *metric topology.* The set $|K|$ with the metric topology is denoted by $|K|_d$.

We shall define another topology on $|K|$. For $s \in K$ the *closed simplex* $|s|$ is defined by

$$|s| = \{\alpha \in |K| \mid \alpha(v) \neq 0 \Rightarrow v \in s\}$$

If s is a q-simplex, $|s|$ is in one-to-one correspondence with the set $\{x \in \mathbf{R}^{q+1} \mid 0 \leq x_i \leq 1, \Sigma\, x_i = 1\}$. Furthermore, the metric topology on $|K|_d$ induces on $|s|$ a topology that makes it a topological space $|s|_d$ homeomorphic to the above compact convex subset of $\mathbf{R}^{q+1}$. If s_1, $s_2 \in K$, then clearly $s_1 \cap s_2$ is either empty (in which case $|s_1| \cap |s_2| = \varnothing$) or a face of s_1 and of s_2 (in which case $|s_1 \cap s_2| = |s_1| \cap |s_2|$). Therefore, in either case $|s_1|_d \cap |s_2|_d$ is a closed set in $|s_1|_d$ and in $|s_2|_d$, and the topology induced on this intersection from $|s_1|_d$ equals the topology induced on it from $|s_2|_d$. It follows from theorem 2.5 in the Introduction that there is a topology on $|K|$ coherent with $\{|s|_d \mid s \in K\}$. This topology will be called the *coherent topology.* The *space* of K, also denoted by $|K|$, is the set $|K|$ with the coherent topology. (What we call here the coherent topology is known in the literature as the weak topology.) Note that $|\bar{s}| = |s|_d$; we shall also use $|s|$ to denote the space $|\bar{s}|$.

Because a subset $A \subset |K|$ is closed (or open) in the coherent topology if and only if $A \cap |s|$ is closed (or open) in $|s|$ for every $s \in K$, we have the following theorem and its corollary.

15 THEOREM *A function f: $|K| \to X$, where X is a topological space, is continuous in the coherent topology if and only if $f \mid |s|$: $|s| \to X$ is continuous for every $s \in K$.* ∎

16 COROLLARY *A function f: $|K| \to X$ is continuous in the coherent topology if and only if $f \mid |K^q|$: $|K^q| \to X$ is continuous for every $q \geq 0$.* ∎

It follows from theorem 15 that the identity map of the set $|K|$ is a continuous map $|K| \to |K|_d$. Note that $L \subset K \Rightarrow |L| \subset |K|$ and $|L|_d$ is a closed subset of $|K|_d$ (which implies that $|L|$ is a closed subset of $|K|$). Furthermore, if $\{L_j\}_{j \in J}$ is a collection of subcomplexes of K, then $\cup\, |L_j| = |\cup\, L_j|$ and $\cap\, |L_j| = |\cap\, L_j|$.

The coherent topology has the following property.

17 THEOREM *For any simplicial complex K, its space $|K|$ is a normal Hausdorff space.*

PROOF Because $|K|_d$ is a Hausdorff space and $i\colon |K| \to |K|_d$ is continuous, $|K|$ is a Hausdorff space. To prove that $|K|$ is normal it suffices to show that if A is a closed subset of $|K|$, any continuous map $f\colon A \to I$ can be continuously extended over $|K|$. By theorem 15, the existence of such an extension of f is equivalent to the existence of an indexed family of continuous maps $\{f_s\colon |s| \to I \mid s \in K\}$ such that

(*a*) If s' is a face of s, then $f_s \mid |s'| = f_{s'}$
(*b*) $f_s \mid (A \cap |s|) = f \mid (A \cap |s|)$

The existence of the family $\{f_s\}$ is proved by induction on dim s. If s is a 0-simplex, $|s|$ is a single point, and either $|s| \in A$, in which case we define $f_s = f \mid |s|$, or $|s| \notin A$, in which case we define f_s arbitrarily.

Let $q > 0$ and assume f_s defined for all simplexes s with dim $s < q$ to satisfy conditions (*a*) and (*b*). Given a q-simplex s, define $f_s'\colon |\dot{s}| \cup (A \cap |s|) \to I$ by the conditions

$$f_s \mid |s'| = f_{s'} \qquad s' \text{ a face of } s$$
$$f_s' \mid (A \cap |s|) = f \mid (A \cap |s|)$$

Because $\{f_{s'}\}_{\dim s' < q}$ satisfies conditions (*a*) and (*b*), f_s' is a continuous map of the closed subset $|\dot{s}| \cup (A \cap |s|)$ of $|s|$ to I. By the Tietze extension theorem, there exists a continuous extension $f_s\colon |s| \to I$ of f_s'. ■

The same technique can be used to prove that $|K|$ is perfectly normal (that is, every closed subset of $|K|$ is the set of zeros of some continuous real-valued function on $|K|$) and paracompact.

For $s \in K$ the *open simplex* $\langle s \rangle \subset |K|$ is defined by

$$\langle s \rangle = \{\alpha \in |K| \mid \alpha(v) \neq 0 \Leftrightarrow v \in s\}$$

Although a closed simplex is a closed set in $|K|$, an open simplex need not be open in $|K|$. However, the open simplex $\langle s \rangle$ is an open subset of $|s|$ because $\langle s \rangle = |s| - |\dot{s}|$. Every point $\alpha \in |K|$ belongs to a unique open simplex (namely, the open simplex $\langle s \rangle$, where $s = \{v \in K \mid \alpha(v) \neq 0\}$). Therefore the open simplexes constitute a partition of $|K|$.

If A is a nonempty subset of $|K|$ that is contained in some closed simplex $|s|$, there is a unique smallest simplex $s \in K$ such that $A \subset |s|$. This smallest simplex is called the *carrier* of A in K. If $A \subset \langle s \rangle$, then the carrier of A is necessarily s. In particular any point α of $|K|$ has as carrier the simplex s such that $\alpha \in \langle s \rangle$.

18 LEMMA Let $A \subset |K|$; *then A contains a discrete subset (in the coherent topology) that consists of exactly one point from each open simplex meeting A.*

PROOF For each $s \in K$ such that $A \cap \langle s \rangle \neq \varnothing$ let $\alpha_s \in A \cap \langle s \rangle$ and let $A' = \{\alpha_s\}$. Because any closed simplex can contain at most a finite subset of A', it follows that every subset of A' is closed in the coherent topology and A' is discrete. ■

Because a compact subset of any topological space can contain no infinite discrete set, we have the following result.

19 COROLLARY *Every compact subset of $|K|$ is contained in the union of a finite number of open simplexes.* ∎

A finite simplicial complex has a compact space. The converse follows from corollary 19.

20 COROLLARY *A simplicial complex K is finite if and only if $|K|$ is compact.* ∎

We establish the following analogue of theorem 15 for homotopies.

21 THEOREM *A function F: $|K| \times I \to X$ is continuous if and only if $F \mid (|s| \times I)$: $|s| \times I \to X$ is continuous for every $s \in K$.*

PROOF Because $|K|$ has the topology coherent with the collection of its closed simplexes, and each closed simplex is a closed compact subset of $|K|$, it follows that $|K|$ is compactly generated. By theorem 2.7 in the Introduction, $|K| \times I$ is also compactly generated. It follows from corollary 19 that every compact subset of $|K| \times I$ is contained in $|L| \times I$ for some finite subcomplex $L \subset K$. Therefore $|K| \times I$ has the topology coherent with the collection $\{|L| \times I \mid L \subset K, L \text{ finite}\}$. It is clear that this topology is identical with the topology coherent with $\{|s| \times I \mid s \in K\}$ (because if L is finite, $|L| \times I$ has the topology coherent with $\{|s| \times I \mid s \in L\}$). ∎

If φ: $K_1 \to K_2$ is a simplicial map, then there is a continuous map $|\varphi|_d$: $|K_1|_d \to |K_2|_d$ defined by

$$|\varphi|_d(\alpha)(v') = \Sigma_{\varphi(v)=v'}\, \alpha(v) \qquad v' \in K_2$$

The same formula defines a continuous map $|\varphi|$: $|K_1| \to |K_2|$, and there is a commutative square

$$\begin{array}{ccc} |K_1| & \to & |K_1|_d \\ {\scriptstyle |\varphi|}\downarrow & & \downarrow{\scriptstyle |\varphi|_d} \\ |K_2| & \to & |K_2|_d \end{array}$$

An easy verification shows that $|\;|$ and $|\;|_d$ are covariant functors from the category of simplicial complexes to the category of topological spaces, and $|K| \to |K|_d$ is a natural transformation between them. These functors can also be regarded as defined on the category of simplicial pairs to the category of pairs of topological spaces.

A *triangulation* (K,f) of a topological space X consists of a simplicial complex K and a homeomorphism f: $|K| \to X$. If X has a triangulation, X is called a *polyhedron.* Similarly a *triangulation* $((K,L), f)$ *of a pair* (X,A) consists of a simplicial pair (K,L) and a homeomorphism f: $(|K|, |L|) \to (X,A)$. If

(X,A) has a triangulation, (X,A) is called a *polyhedral pair.* In general, a given polyhedron will have triangulations (K_1,f_1) and (K_2,f_2), for which K_1 and K_2 are not isomorphic simplicial complexes.

Following are some examples.

22 For any $n \geq 1$, $(E^{n+1},S^n)$is homeomorphic to $(|\bar{s}|,|\dot{s}|)$, where s is an $(n+1)$-simplex. Therefore (E^{n+1},S^n) is a polyhedral pair.

23 Given that K is the simplicial complex of example 8 and $f\colon |K| \to \mathbf{R}$ is defined so that $f(|\{n\}|) = n$ and $f \mid |\{n, n+1\}|$ is a homeomorphism of $|\{n, n+1\}|$ onto the closed interval $[n, n+1]$, then (K,f) is a triangulation of $\mathbf{R}$, and $\mathbf{R}$ is a polyhedron.

24 For $n \geq 1$, given that K is the simplicial complex of example 9 and $f\colon |K| \to \mathbf{R}^n$ is defined by the equation $(f(\alpha))_i = \sum_{x \in Z^n} \alpha(x)(x)_i$, then (K,f) is a triangulation of $\mathbf{R}^n$, and $\mathbf{R}^n$ is a polyhedron.

Given a vertex $v \in K$, its *star* is defined by

$$\operatorname{st} v = \{\alpha \in |K| \mid \alpha(v) \neq 0\}$$

Because $\alpha \to \alpha(v)$ is a continuous map from $|K|_d$ to I, st v is open in $|K|_d$, and hence also in $|K|$. It is immediate from the definition that

$$\begin{aligned} \alpha \in \operatorname{st} v &\Leftrightarrow \text{carrier } \alpha \text{ has } v \text{ as vertex} \\ &\Leftrightarrow \alpha \in \langle s \rangle \qquad \text{where } s \text{ has } v \text{ as vertex} \end{aligned}$$

Therefore st $v = \bigcup\{\langle s \rangle \mid v \text{ is vertex of } s\}$.

25 LEMMA *Let $L \subset K$ and let $v_0, v_1, \ldots, v_q$ be vertices of K. Then $v_0, v_1, \ldots, v_q$ are vertices of a simplex of L if and only if*

$$\bigcap_{0 \leq i \leq q} \operatorname{st} v_i \cap |L| \neq \varnothing$$

PROOF If there is a simplex $s \in L$ with vertices $v_0, \ldots, v_q$, then $\langle s \rangle \subset \operatorname{st} v_i$ for every i, and $\langle s \rangle \subset |L|$. Therefore $\bigcap \operatorname{st} v_i \cap |L| \neq \varnothing$. Conversely, if $\bigcap \operatorname{st} v_i \cap |L| \neq \varnothing$, let $\alpha \in \bigcap \operatorname{st} v_i \cap |L|$. Then $\alpha(v_i) \neq 0$ for $0 \leq i \leq q$, and carrier α is a simplex s of L whose vertices include $v_0, \ldots, v_q$. Then the set $\{v_0, \ldots, v_q\}$ is a face of s and must belong to L, because L is a complex. ■

This yields the following relation between K and the open covering of $|K|$ of vertex stars.

26 THEOREM *Let $\mathfrak{U} = \{\operatorname{st} v \mid v \in K\}$. The vertex map φ from K to $K(\mathfrak{U})$ defined by $\varphi(v) = \operatorname{st} v$ is a simplicial isomorphism $\varphi\colon K \approx K(\mathfrak{U})$, and for any $L \subset K$, $\varphi \mid L\colon L \approx K_{|L|}(\mathfrak{U})$.* ■

2 LINEARITY IN SIMPLICIAL COMPLEXES

The linear structure in the set of all functions from any set to $\mathbf{R}$ defines linearity in the space of a simplicial complex. This section is devoted to a study of such linearity. We show that a closed simplex $|s|$ is homeomorphic to the cone with base $|\dot{s}|$. This implies that a closed simplex can be parametrized by "polar coordinates," which are convenient for the construction of maps. We use them to prove that a polyhedral pair has the homotopy extension property with respect to any space.

We also consider linear imbeddings in euclidean space of the space of a simplicial complex; this entails a discussion of locally finite simplicial complexes. Such complexes are characterized by the property that their spaces are locally compact or the equivalent property that the coherent and metric topologies coincide on their spaces.

Let K be a simplicial complex and let $\alpha_1, \ldots, \alpha_p$ be points of a closed simplex $|s|$. Given real numbers $t_1, \ldots, t_p$ such that $0 \le t_i \le 1$ for $i = 1, \ldots, p$ and such that $\Sigma t_i = 1$, the function $\alpha = \Sigma t_i \alpha_i$ is again a point of $|s|$. Therefore each closed simplex has a linear structure such that convex combinations of its points are again points of the closed simplex. Conversely, if $\alpha = \Sigma t_i \alpha_i$ has a simplex s as carrier (so that $\alpha \in \langle s \rangle$), then each $\alpha_i \in |s|$. Therefore we have the following lemma.

1 LEMMA *A convex combination of points of $|K|$ is again a point of $|K|$ if and only if the points all lie in some closed simplex.* ∎

We shall find it convenient to identify the vertices of K with their characteristic functions. That is, if v is a vertex of K, we regard v as also being the function from vertices $v' \in K$ defined by

$$v(v') = \begin{cases} 0 & v \neq v' \\ 1 & v = v' \end{cases}$$

If $\alpha \in |K|$, then we can write $\alpha = \Sigma_{v \in K}\, \alpha(v) v$, the sum on the right being a convex combination of points of $|K|$.

Let X be a topological space which is a subset of some real vector space. We assume that X has a topology coherent with its intersections with finite-dimensional subspaces each such intersection being topologized as a subspace of the finite-dimensional topological linear space in which it lies. For example, X is euclidean space or X is the space of a simplicial complex. A continuous map $f\colon |K| \to X$ is said to be *linear on* K if it is linear in terms of barycentric coordinates. That is, f is linear if for every $\alpha \in |K|$, $\Sigma_{v \in K}\, \alpha(v) f(v)$ is a point of X and

$$f(\alpha) = \Sigma_{v \in K}\, \alpha(v) f(v)$$

It is then clear that a linear map is uniquely determined by the vertex map f_0 from vertices of K to X such that $f_0(v) = f(v)$. Conversely, a vertex map f_0 from vertices of K to X may be extended to a linear map $f\colon |K| \to X$ if and only if for every simplex $s \in K$ all convex combinations of elements in $f_0(s)$ lie in X.

If $\varphi\colon K_1 \to K_2$ is a simplicial map, then the definition of $|\varphi|$ shows that

$$|\varphi|(\alpha) = \Sigma\, \alpha(v)|\varphi|(v)$$

Therefore $|\varphi|$ is linear.

Let X be a topological space. The *cone* $X * w$ with *base* X and *vertex* w is defined to be the mapping cylinder of the constant map $X \to w$. The points of $X * w$ are parametrized by $[x,t]$ with $x \in X$ and $t \in I$, where $x \in X$ is identified with $[x,0]$ and $[x,1]$ is identified with w for all $x \in X$. Because w is a strong deformation retract of $X * w$, a cone is contractible.

2 LEMMA *For any simplex s of K the cone* $|\dot{s}| * w$ *is homeomorphic to* $|s|$.

PROOF Choose a point $w_0 \in \langle s \rangle$ and define a map $f\colon |\dot{s}| * w \to |s|$ by $f([\alpha,t]) = tw_0 + (1-t)\alpha$. Then f is continuous (because the linear operations in $|s|$ are continuous). To show that f is injective, assume $f([\alpha,t]) = f([\beta,t'])$ for $\alpha, \beta \in |\dot{s}|$ and $t, t' \in I$. Then

$$tw_0 + (1-t)\alpha = t'w_0 + (1-t')\beta$$

Let s have vertices $v_0, v_1, \ldots, v_q$ and suppose that $\alpha = \Sigma\alpha_i v_i$, $\beta = \Sigma\beta_i v_i$, and $w_0 = \Sigma\, \gamma_i v_i$. Because $\alpha, \beta \in |\dot{s}|$, there is j such that $\alpha_j = 0$ and there is k such that $\beta_k = 0$. Then

$$t\gamma_j = t'\gamma_j + (1-t')\beta_j \qquad \text{and} \qquad (t-t')\gamma_j = (1-t')\beta_j$$

Because $\gamma_j \neq 0$, $t \geq t'$. Similarly, $t\gamma_k + (1-t)\alpha_k = t'\gamma_k$ and so $t' \geq t$. Therefore $t = t'$. It follows then that $(1-t)\,\alpha = (1-t)\beta$, and if $t \neq 1$, $\alpha = \beta$. Therefore either $t = t'$ and $\alpha = \beta$ or $t = t' = 1$. In either case $[\alpha,t] = [\beta,t']$, and f is injective.

We now show that f is surjective. Clearly, $f([\alpha,0]) = \alpha$ and $f([\alpha,1]) = w_0$, and so f maps onto $|\dot{s}|$ and w_0. To show that every point of $\langle s \rangle - w_0$ is on a unique line segment from w_0 to some point of $|\dot{s}|$, let $\alpha \in \langle s \rangle$, with $\alpha \neq w_0$, and suppose that $\alpha = \Sigma\alpha_i v_i$. Consider the function $\varphi(t') = (1+t')\alpha - t'w_0$. $\varphi(0) = \alpha \in \langle s \rangle$, and as t' increases, the barycentric coordinates of $\varphi(t')$ change continuously. Because $\alpha \neq w_0$, there is some i such that $\alpha_i < \gamma_i$. Therefore

$$\varphi(t')(v_i) = \alpha_i - t'(\gamma_i - \alpha_i)$$

is a monotonically decreasing function of t'. By continuity, there exists a unique $t' > 0$ such that $\varphi(t')(v_i) = 0$. Hence there exists a $t_0' > 0$ which is the smallest t' for which $\varphi(t_0')(v_i) = 0$ for any $0 \leq i \leq q$. Then $\varphi(t_0') \in |\dot{s}|$ and

$$\alpha = \frac{t_0'}{1+t_0'}w_0 + \frac{1}{1+t_0'}\varphi(t_0')$$

shows that $\alpha = f([\varphi(t_0'), t_0'/(1+t_0')])$, and f is surjective.

Because f is a continuous bijection from a compact space to a Hausdorff space, it is a homeomorphism. ■

The *barycenter* $b(s)$ of the simplex $s = \{v_0, v_1, \ldots, v_q\}$ is defined to be the point

$$b(s) = \Sigma_{0 \le i \le q} \frac{1}{q+1} v_i$$

Clearly, $b(s) \in \langle s \rangle$, and so the carrier of $b(s)$ is s. By lemma 2, $|s|$ is homeomorphic to $|\dot{s}| * w$ in such a way that w corresponds to $b(s)$. If $\alpha \in |\dot{s}|$ and $t \in I$, the point $tb(s) + (1-t)\alpha$ of $|s|$ will be parametrized by *polar coordinates* $[\alpha,t]$, where $[\alpha,t]$ denotes the point of $|\dot{s}| * w$ corresponding to the given point of $|s|$. Then $[\alpha,0] = \alpha$ and $[\alpha,1] = b(s)$ for all $\alpha \in |\dot{s}|$. We use polar coordinates for the following homotopy.

3 LEMMA *For any simplex s, $|s| \times 0 \cup |\dot{s}| \times I$ is a strong deformation retract of $|s| \times I$.*

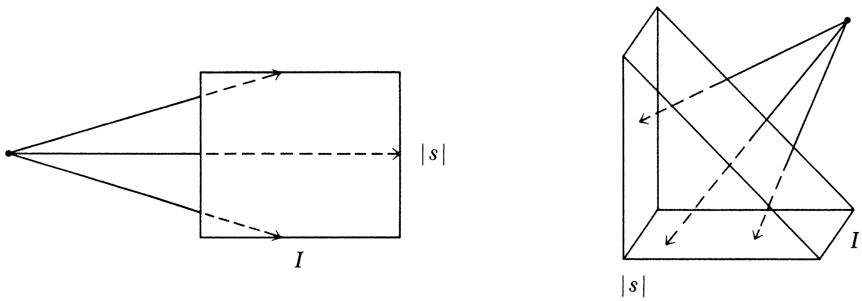
PROOF If s is a 0-simplex, $|\dot{s}| = \varnothing$ and we know the point $|s| \times 0$ is a strong deformation retract of the closed interval $|s| \times I$. If $\dim s > 0$, we define a deformation retraction

$$F: |s| \times I \times I \to |s| \times I$$

to $|s| \times 0 \cup |\dot{s}| \times I$ by the formula in polar coordinates

$$F([\alpha,t], t', t'') = \begin{cases} \left(\left[\alpha, (1-t'')t + \dfrac{t''(2t-t')}{2-t'}\right], (1-t'')t'\right) & t' \le 2t \\ \left([\alpha, (1-t'')t], (1-t'')t' + \dfrac{t''(t'-2t)}{1-t}\right) & 2t \le t' \end{cases}$$

and diagram it for the cases of a 1-simplex and a 2-simplex:



■

4 COROLLARY *For any subcomplex $L \subset K$ the subspace $|K| \times 0 \cup |L| \times I$ is a strong deformation retract of $|K| \times I$.*

PROOF Let $X^n = |K| \times 0 \cup |K^n \cup L| \times I$ for $n \ge -1$. We first show that for each $n \ge 0$ the space X^{n-1} is a strong deformation retract of X^n. For each n-simplex $s \in K - L$ let $F_s: |s| \times I \times I \to |s| \times I$ be a strong deformation retraction of $|s| \times I$ to $|s| \times 0 \cup |\dot{s}| \times I$ (which exists, by lemma 3). For $n \ge 0$ define a map

$$F_n: X^n \times I \to X^n$$

by the conditions

$$F_n \mid |s| \times I \times I = F_s \qquad \text{for an } n\text{-simplex } s \in K - L$$
$$F_n(x,t) = x \qquad x \in X^{n-1},\ t \in I$$

Then F_n is well-defined and continuous (because for every simplex s the restriction $F_n \mid |s| \times I \times I$ is continuous), and F_n is a strong deformation retraction of X^n to X^{n-1}.

Let $f_n\colon X^n \to X^{n-1}$ be the retraction defined by $f_n(x) = F_n(x,1)$ for $x \in X^n$. Let $a_n = 1/n$ for $n \geq 1$, and define $G_n\colon X^n \times I \to X^n$ by induction on n so that

$$G_0(x,t) = \begin{cases} x & 0 \leq t \leq a_2 \\ F_0\left(x, \dfrac{t - a_2}{1 - a_2}\right) & a_2 \leq t \leq 1 \end{cases}$$

and for $n \geq 1$

$$G_n(x,t) = \begin{cases} x & 0 < t \leq a_{n+2} \\ F_n\left(x, \dfrac{t - a_{n+2}}{a_{n+1} - a_{n+2}}\right) & a_{n+2} \leq t \leq a_{n+1} \\ G_{n-1}(f_n(x),t) & a_{n+1} \leq t \leq 1 \end{cases}$$

By induction on n, it is easily verified that G_n is a strong deformation retraction of X^n to X^{-1} such that $G_n \mid X^{n-1} \times I = G_{n-1}$. Therefore there is a map

$$G\colon |K| \times I \times I \to |K| \times I$$

such that $G \mid X^n \times I = G_n$. Then G is a strong deformation retraction of $|K| \times I$ to $|K| \times 0 \cup |L| \times I$. ∎

5 COROLLARY *A polyhedral pair has the homotopy extension property with respect to any space.*

PROOF It suffices to show that if $L \subset K$, then $(|K|, |L|)$ has the homotopy extension property with respect to any space Y. Given $g\colon |K| \to Y$ and $G\colon |L| \times I \to Y$ such that $G(\alpha,0) = g(\alpha)$ for $\alpha \in |L|$, let $f\colon |K| \times 0 \cup |L| \times I \to Y$ be defined by $f(\alpha,0) = g(\alpha)$ for $\alpha \in |K|$ and $f(\alpha,t) = G(\alpha,t)$ for $\alpha \in |L|$ and $t \in I$. Because $|L|$ is closed in $|K|$, f is continuous. By corollary 4, $|K| \times 0 \cup |L| \times I$ is a retract of $|K| \times I$. Therefore f can be extended to a continuous map $F\colon |K| \times I \to Y$. Then $F(\alpha,0) = g(\alpha)$ for $\alpha \in |K|$ and $F \mid |L| \times I = G$. ∎

Let us now consider linear imbeddings of $|K|$ in euclidean space.

6 LEMMA *A linear map $f\colon |s| \to \mathbf{R}^n$ is an imbedding if and only if it maps the vertex set of s to an affinely independent set in $\mathbf{R}^n$.*

PROOF Let $f(v_i) = p_i$, where $s = \{v_i\}$. We show that the set $\{p_i\}$ is affinely dependent if and only if f is not injective. $\{p_i\}$ is affinely dependent if and only if there exist α_i not all zero such that $\Sigma\alpha_i p_i = 0$ and $\Sigma\alpha_i = 0$. Assume the points p_i enumerated so that $\alpha_i \geq 0$ for $i \leq j_0$ and $\alpha_i < 0$ for $i > j_0$.

Then $\sum_{i \le j_0} \alpha_i p_i = \sum_{i > j_0} (-\alpha_i) p_i$. If $a = \sum_{i \le j_0} \alpha_i = \sum_{i > j_0} -\alpha_i$, then $\sum_{i \le j_0} (\alpha_i/a) p_i = \sum_{i > j_0} (-\alpha_i/a) p_i$. It follows from the linearity of f that $f(\sum_{i \le j_0} (\alpha_i/a) v_i) = f(\sum_{i > j_0} (-\alpha_i/a) v_i)$, showing that f is not injective.

Conversely, if f is not injective, then $f(\sum \alpha_i v_i) = f(\sum \beta_i v_i)$, where $\alpha_{j_0} \ne \beta_{j_0}$ for some j_0. Then $\sum(\alpha_i - \beta_i) p_i = 0$ and $\sum(\alpha_i - \beta_i) = 0$. Because $\alpha_{j_0} - \beta_{j_0} \ne 0$, the set $\{p_i\}$ is affinely dependent. ∎

A simplicial complex K is said to be *locally finite* if every vertex v of K belongs to only finitely many simplexes of K.

7 LEMMA *If K is locally finite, every point of $|K|_d$ has a neighborhood of the form $|L|_d$, where L is a finite subcomplex of K.*

PROOF Let $\alpha \in |K|_d$. Then $\alpha \in \text{st } v$ for some vertex v of K. Because v is a vertex of only finitely many simplexes $\{s_i\}$ of K, st v is contained in the compact set $\bigcup |s_i|$. Let $L = \{s \in K \mid s \text{ is a face of } s_i \text{ for some } i\}$. Then L is a finite subcomplex of K, and $\alpha \in \text{st } v \subset |L|_d$. ∎

8 THEOREM *For a simplicial complex K, the following are equivalent:*

(*a*) *K is locally finite.*
(*b*) *$|K|$ is locally compact.*
(*c*) *$|K| \to |K|_d$ is a homeomorphism.*
(*d*) *$|K|$ is metrizable.*
(*e*) *$|K|$ satisfies the first axiom of countability.*

PROOF $(a) \Rightarrow (b)$. By lemma 7, if α is a point of $|K|_d$, there is a finite subcomplex $L \subset K$ such that α is in the interior of $|L|_d$. Then α is in the interior of $|L|$ in $|K|$. Therefore $|L|$ is a compact neighborhood of α in $|K|$.

$(b) \Rightarrow (c)$. To show that $|K| \to |K|_d$ is an open map, let U be an open subset of $|K|$ with compact closure $\bar U$ in $|K|$. It suffices to show that U is open in $|K|_d$. Because $\bar U$ is compact, there is a finite subcomplex $L \subset K$ such that $\bar U \subset |L|$ (by corollary 3.1.19). Let K_1 be the subcomplex of K defined by

$$K_1 = \{s \in K \mid |s| \cap U = \varnothing\}$$

If $s \in K - K_1$, then $|s| \cap U$ is a nonempty open subset of $|s|$. Therefore $\langle s \rangle \cap U \ne \varnothing$ and $\langle s \rangle \cap |L| \ne \varnothing$. The fact that the open simplexes of K form a partition of K implies that $s \in L$, and we have shown that $K = K_1 \cup L$. Now, $|K|_d - |K_1|_d$ is an open subset of $|K|_d$. Because L is finite, $|L| \to |L|_d$ is a homeomorphism. Therefore U is open in $|L|_d$, and so it is open in $|L|_d - |K_1|_d$. Because $|L|_d - |K_1|_d = |K|_d - |K_1|_d$, U is open in $|K|_d$.

$(c) \Rightarrow (d)$. Because $|K|_d$ is metrizable, if $|K|$ and $|K|_d$ are homeomorphic, then $|K|$ is also metrizable.

$(d) \Rightarrow (e)$. Every metrizable space satisfies the first axiom of countability.

$(e) \Rightarrow (a)$. Assume that K is not locally finite and let v be a vertex of an infinite set of simplexes $\{s_i\}_{i=1,2,\ldots}$ of K. Assume that v has a countable base of neighborhoods $\{U_i\}_{i=1,2,\ldots}$ in $|K|$. Without loss of generality, we may assume $U_i \supset U_{i+1}$ for all $i \geq 1$. For each i, $\langle s_i \rangle \cap U_i \neq \varnothing$, because v, being a vertex of s_i, is in the closure of $\langle s_i \rangle$. Let $\alpha_i \in \langle s_i \rangle \cap U_i$. Then the sequence $\{\alpha_i\}$ has v as a limit point (because each U_i contains all α_j with $j \geq i$), but in the coherent topology the set $\{\alpha_i\}$ is discrete, because it meets every closed simplex $|s|$ in a finite set. ■

A *realization* of a simplicial complex K in $\mathbf{R}^n$ is a linear imbedding of $|K|$ in $\mathbf{R}^n$. The following theorem characterizes those complexes K which have realizations in some euclidean space.

9 THEOREM *If K has a realization in* $\mathbf{R}^n$, *then K is countable and locally finite, and* $\dim K \leq n$. *Conversely, if K is countable and locally finite, and* $\dim K \leq n$, *then K has a realization as a closed subset in* $\mathbf{R}^{2n+1}$.

PROOF Let $f\colon |K| \to \mathbf{R}^n$ be a linear imbedding. If K is uncountable, it follows from lemma 3.1.18 that $|K|$ contains an uncountable discrete set A'. Then $f(A')$ is an uncountable discrete subset of $\mathbf{R}^n$, which is impossible because $\mathbf{R}^n$ is separable. Therefore K is countable. Clearly $|K|$ is metrizable and, by theorem 8, K is locally finite. It follows from lemma 6 and theorem 5.3 in the Introduction that $\dim K \leq n$.

To prove the converse statement, let $\{p_i\}$ be a sequence of points in $\mathbf{R}^{2n+1}$ such that

(*a*) Every set of $2n + 2$ of the points p_i is affinely independent.
(*b*) If C is any compact subset of $\mathbf{R}^{2n+1}$, there exists j such that C is disjoint from the convex subset of $\mathbf{R}^{2n+1}$ generated by the set $\{p_i \mid i \geq j\}$.

For example, let $H_1 \supset H_2 \supset \cdots$ be a decreasing sequence of closed half-spaces of $\mathbf{R}^{2n+1}$ such that $\bigcap H_i = \varnothing$, and assuming p_i defined for $i < q$, inductively choose p_q to be a point of H_q not lying on any of the finite number of affine varieties determined by $2n + 1$ or fewer points of the set $\{p_i \mid 1 \leq i \leq q - 1\}$.)

Assume that K is countable and locally finite and $\dim K \leq n$, and let $\{v_i\}_{i=1,2,\ldots}$ be an enumeration of the vertices of K. Define $f\colon |K| \to \mathbf{R}^{2n+1}$ to be the linear map such that $f(v_i) = p_i$. Because of condition (*a*), it follows that for any $s \in K$, $f \mid |s|$ is a linear imbedding of $|s|$ in $|K|$, and if s and $s' \in K$, then

$$f(|s| \cap |s'|) = f(|s|) \cap f(|s'|)$$

Therefore f is injective. Because of condition (*b*), if C is any compact subset of $\mathbf{R}^{2n+1}$, there is j such that $f^{-1}(C) \subset \bigcup \{\mathrm{st}\, v_i \mid i \leq j\}$. Since K is locally finite, this implies that $f^{-1}(C) \subset |L|$ for some finite subcomplex $L \subset K$. Therefore $f^{-1}(C)$ is compact in $|K|$. If A is closed in $|K|$ and C is compact in $\mathbf{R}^{2n+1}$, then $f(A) \cap C = f(A \cap f^{-1}(C))$ is closed in C [because $A \cap f^{-1}(C)$ is a closed subset of the compact subset $f^{-1}(C)$ of $|K|$ and $f \mid f^{-1}(C)$ is a homeomorphism of

$f^{-1}(C)$ to $f(f^{-1}(C))$]. Therefore f is a closed map and is a linear imbedding of $|K|$ as a closed subset in $\mathbf{R}^{2n+1}$. ∎

3 SUBDIVISION

Our main interest in simplicial complexes is in the polyhedra they describe. To study a polyhedron it is important to consider its different triangulations and their interrelationships. This section is devoted to proving the existence of "small" triangulations of a polyhedron, which are used in the next section in proving that arbitrary continuous maps between polyhedra can be approximated by simplicial maps.

Let K be a simplicial complex. A *subdivision* of K is a simplicial complex K' such that

(*a*) The vertices of K' are points of $|K|$.
(*b*) If s' is a simplex of K', there is some simplex s of K such that $s' \subset |s|$ (that is, s' is a finite nonempty subset of $|s|$).
(*c*) The linear map $|K'| \to |K|$ mapping each vertex of K' to the corresponding point of $|K|$ is a homeomorphism.

Note that conditions (*a*) and (*b*) assert that every simplex s' of K' has a carrier $s \in K$. If K' is a subdivision of K, we identify $|K'|$ and $|K|$ by the linear homeomorphism of condition (*c*). The following fact is immediate from the definition.

1 *Any subdivision of a subdivision of K is itself a subdivision of K.* ∎

The next fact is also true (but somewhat more difficult to prove).

2 *If K' and K'' are subdivisions of K, there is a subdivision K''' of K that is a subdivision of K' and of K''.* ∎

Thus, statements 1 and 2 assert that the subdivisions of K form a directed set with respect to the partial ordering defined by the relation of subdivision.

3 LEMMA *Let K and K' be simplicial complexes satisfying conditions* (*a*) *and* (*b*). *If $s \in K$ is the carrier of $s' \in K'$, then $\langle s' \rangle \subset \langle s \rangle$.*

PROOF Let $v_0', \ldots, v_p'$ be the vertices of s' and let $v_0, \ldots, v_q$ be the vertices of the carrier s of s'. Because $s' \subset |s|$, for $0 \le i \le p$, $v_i' = \Sigma \alpha_{ij} v_j$. Because s is the smallest such simplex, for $0 \le j \le q$ there exists $0 \le i \le p$ such that $\alpha_{ij} \ne 0$. Let $\beta \in \langle s' \rangle$. Then

$$\beta = \sum_i \beta_i v_i' = \sum_j \left(\sum_i \beta_i \alpha_{ij}\right) v_j$$

and because $\beta_i > 0$ for all i, $\sum_i \beta_i \alpha_{ij} > 0$ for all j. Therefore $\beta \in \langle s \rangle$ and $\langle s' \rangle \subset \langle s \rangle$. ∎

4 THEOREM *Let K' and K be simplicial complexes satisfying conditions (a) and (b). Then K' is a subdivision of K if and only if for $s \in K$ the set $\{\langle s'\rangle \mid s' \in K', \langle s'\rangle \subset \langle s\rangle\}$ is a finite partition of $\langle s\rangle$.*

PROOF Assume that K' and K satisfy conditions (a) and (b) and the condition that $\{\langle s'\rangle \mid s' \in K', \langle s'\rangle \subset \langle s\rangle\}$ is a finite partition of $\langle s\rangle$ for $s \in K$. Because any simplex $s \in K$ has only a finite number of faces, it follows that

$$K'(s) = \{s' \in K' \mid \text{there exists a face } s_1 \text{ of } s \text{ such that } \langle s'\rangle \subset \langle s_1\rangle\}$$

is a finite subcomplex of K', and the linear map $h_s\colon |K'(s)| \to |s|$ that maps each vertex of $K'(s)$ to itself is a homeomorphism. Therefore there is a continuous map $g\colon |K| \to |K'|$ such that $g \mid |s| = h_s^{-1}$ for $s \in K$, which is an inverse of the linear map $h\colon |K'| \to |K|$. Therefore h is a homeomorphism, and K' and K satisfy condition (c).

Conversely, if K' is a subdivision of K, then $\{s' \mid s' \in K'\}$ is a partition of $|K'| = |K|$. For $s \in K$, consider the sets $\langle s'\rangle \cap \langle s\rangle$ for $s' \in K'$. By lemma 3, either $\langle s'\rangle \cap \langle s\rangle = \varnothing$ or $\langle s'\rangle \subset \langle s\rangle$. Therefore $\{\langle s'\rangle \mid s' \in K', \langle s'\rangle \subset \langle s\rangle\}$ is a partition of $\langle s\rangle$. Because $|s|$ is compact, it follows from corollary 3.1.19 that this set is a finite partition of $\langle s\rangle$. ■

We use this result to show that any subdivision of K simultaneously subdivides every subcomplex of K.

5 COROLLARY *Let K' be a subdivision of K and let L be a subcomplex of K. There is a unique subcomplex L' of K' which is a subdivision of L.*

PROOF If L' is a subcomplex of K' that is a subdivision of L, then $L' = \{s' \in K' \mid \langle s'\rangle \subset |L|\}$, which proves the uniqueness of L'. To prove the existence of L', we prove that $\{s' \in K' \mid \langle s'\rangle \subset |L|\}$ has the desired properties. It is clear that this set is a subcomplex L' of K' and that L' and L satisfy conditions (a) and (b) above. We use theorem 4 to show that L' is a subdivision of L. If $s \in L$, by theorem 4 the set $\{\langle s'\rangle \mid s' \in K', \langle s'\rangle \subset \langle s\rangle\}$ is a finite partition of $\langle s\rangle$. By definition of L',

$$\{\langle s'\rangle \mid s' \in K', \langle s'\rangle \subset \langle s\rangle\} = \{\langle s'\rangle \mid s' \in L', \langle s'\rangle \subset \langle s\rangle\}$$

Therefore, by theorem 4, L' is a subdivision of L. ■

The subdivision L' of L in corollary 5 is called the *subdivision of L induced by K'* and is denoted by $K' \mid L$.

From the definition of subdivision two facts are immediate.

6 *If $f\colon |K| \to X$ is linear on K and K' is a subdivision of K, then f is also linear on K'.* ■

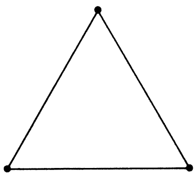
7 *If $((K,L), f)$ is a triangulation of (X,A) and K' is a subdivision of K, then $((K',K' \mid L), f)$ is also a triangulation of (X,A).* ■

For any simplicial complex we construct a particular subdivision, called the barycentric subdivision. For this we need the following lemma, which shows how to extend a subdivision of $\dot{s}$ to a subdivision of $\bar{s}$ for any simplex s.

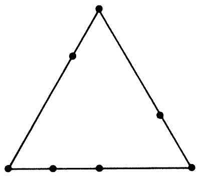
8 LEMMA *Let s be a simplex of some complex and let K' be a subdivision of $\dot{s}$. For any $w_0 \in \langle s \rangle$, $K' * w_0$ is a subdivision of $\bar{s}$.*

PROOF In the statement of lemma 8, w_0 is regarded as a simplicial complex having a single vertex and $K' * w_0$ is the join defined in example 3.1.7. It is clear that $K' * w_0$ satisfies requirements (*a*) and (*b*) for a subdivision of $\bar{s}$. It follows from lemma 3.2.2 that any point of $|s|$ either equals w_0, belongs to $|\dot{s}|$, or belongs to a unique open simplex of the form $\langle s' \cup \{w_0\} \rangle$, where $s' \in K'$. Therefore the open simplexes of $|K' * w_0|$ constitute a finite partition of $|s|$, and by theorem 4, $K' * w_0$ is a subdivision of $\bar{s}$. ∎

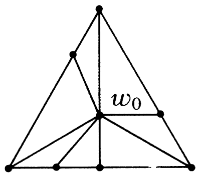
The subdivision of $\bar{s}$ obtained by applying lemma 8 is pictured below for a 2-simplex s.



s = triangle and its faces



K' = pictured subdivision of the boundary of the triangle



$K' * w_0$ = pictured triangles and their faces

We are now ready to prove the existence of the barycentric subdivision. Let K be a simplicial complex. We define sd K to be the simplicial complex whose vertices are the barycenters of the simplexes of K and whose simplexes are finite nonempty collections of barycenters of simplexes which are totally ordered by the face relation in K. Thus the simplexes of sd K are finite sets $\{b(s_0), \ldots, b(s_q)\}$ such that s_{i-1} is a face of s_i for $i = 1, \ldots, q$. We shall always assume the vertices of a simplex of sd K to be enumerated in this order.

It is clear that sd K is a simplicial complex and that if L is a subcomplex of K, then sd L is a subcomplex of sd K. Furthermore, if $b(s_q)$ is the last vertex of a simplex $s' \in$ sd K, then $s' \subset |s_q|$, and since s_q is the carrier of $b(s_q)$, s_q is the carrier of s'. Therefore sd K and K satisfy conditions (*a*) and (*b*).

9 THEOREM sd *K is a subdivision of K.*

PROOF We show that sd K and K satisfy the hypotheses of theorem 4. If $s \in K$, then, by lemma 3 and the remarks above,

$$\begin{aligned}\{s' \in \operatorname{sd} K \mid \langle s' \rangle \subset \langle s \rangle\} &= \{s' \in \operatorname{sd} K \mid \text{last vertex of } s' = b(s)\}\\ &= \{s' \in \operatorname{sd} \bar{s} \mid \langle s' \rangle \subset \langle s \rangle\}\end{aligned}$$

Therefore we need only show that sd $\bar{s}$ is a subdivision of $\bar{s}$ for any $s \in K$. We do this by induction on dim s. If dim $s = 0$, sd $\bar{s} = \bar{s}$ is a subdivision of $\bar{s}$. For $q > 0$, assume that sd $\bar{s}_1$ is a subdivision of $\bar{s}_1$ for every simplex s_1 with dim $s_1 < q$, and let s be a q-simplex. By the inductive assumption, sd $\dot{s}$ is a subdivision of $\dot{s}$. The definition of the barycentric subdivision shows that sd $\bar{s}$ = sd $\dot{s} * b(s)$. By lemma 8, this is a subdivision of $\bar{s}$. ∎

The subdivision sd K is called the *barycentric subdivision of* K. The *iterated barycentric subdivisions* $\mathrm{sd}^n K$ are defined for $n \geq 0$ inductively, so that

$$\mathrm{sd}^0 K = K$$
$$\mathrm{sd}^n K = \mathrm{sd}\,(\mathrm{sd}^{n-1} K) \qquad n \geq 1$$

10 LEMMA *If L is a subcomplex of K,* sd L *is a full subcomplex of* sd K.

PROOF Let $\{b(s_0), \ldots ,b(s_q)\}$ be a simplex of sd K all of whose vertices belong to sd L. Then s_{i-1} is a face of s_i for $i = 1, \ldots , q$ and each $s_i \in L$. Therefore $\{b(s_0), \ldots ,b(s_q)\} \in \mathrm{sd}\, L$. ∎

11 COROLLARY *Let (X,A) be a polyhedral pair. Then A is a strong deformation retract of some neighborhood of A in X.*

PROOF Because of statement 7 and lemma 10, it suffices to consider the case $(X,A) = (|K|,|L|)$, where L is a full subcomplex of K. Let N be the largest subcomplex of K disjoint from L. We prove that $|L|$ is a strong deformation retract of $|K| - |N|$. If $\alpha \in |K| - |N|$, then, by lemma 3.1.10, either $\alpha \in |L|$ or there exist vertices $v_0, \ldots , v_p \in L$ and vertices $v_{p+1}, \ldots , v_q \in N$, with $0 \leq p$ and $p + 1 \leq q$, such that $\alpha \in \langle v_0, \ldots ,v_q\rangle$. In the latter case, $\alpha = \sum_{0\leq i\leq q}\alpha_i v_i$, with $\alpha_i > 0$, and we define $a = \sum_{0\leq i\leq p}\alpha_i$. Then $0 < a < 1$ and we let $\alpha_i' = \alpha_i/a$ for $0 \leq i \leq p$ and $\alpha_i'' = \alpha_i/(1 - a)$ for $p + 1 \leq i \leq q$. Then $\alpha = a\alpha' + (1 - a)\alpha''$, where $\alpha' = \sum_{0\leq i\leq p}\alpha_i' v_i$ is in $|L|$ and $\alpha'' = \sum_{p+1\leq i\leq q}\alpha_i'' v_i$ is in $|N|$. A strong deformation retraction $F\colon (|K| - |N|) \times I \to |K| - |N|$ of $|K| - |N|$ to $|L|$ is defined by

$$F(\alpha,t) = \begin{cases} \alpha & \alpha \in |L|,\ t \in I \\ t\alpha' + (1 - t)\alpha & \alpha \in |K| - (|N| \cup |L|),\ t \in I \end{cases}$$

F is continuous because $F \mid |L| \times I$ is continuous, and for any simplex of K of the form $s' \cup s''$, where $s' \in L$ and $s'' \in N$, $F \mid [|s' \cup s''| \cap (|K| - |N|)] \times I$ is continuous. ∎

Let X be a polyhedron and let $\mathcal{U}$ be an open covering of X. A triangulation (K,f) of X is said to be *finer* than $\mathcal{U}$ if for every vertex $v \in K$ there is $U \in \mathcal{U}$ such that $f(\mathrm{st}\, v) \subset U$. A simplicial complex K is said to be *finer* than an open covering $\mathcal{U}$ of $|K|$ if the triangulation $(K,1_{|K|})$ of $|K|$ is finer than $\mathcal{U}$ (that is, for each vertex $v \in K$ there is $U \in \mathcal{U}$ such that $\mathrm{st}\, v \subset U$). We show that if $\mathcal{U}$ is any open covering of a compact polyhedron, there are triangulations finer than $\mathcal{U}$.

A metric on $|K|$ is said to be *linear on* K if it is induced from the norm in $\mathbf{R}^n$ by a realization of K in $\mathbf{R}^n$. Any finite simplicial complex has linear metrics, and if K' is any subdivision of K, a metric that is linear on $|K|$ is also linear on $|K'|$.

12 LEMMA *Given a metric linear on an m-simplex s, then for any $s' \in \mathrm{sd}\, \bar{s}$*

$$\mathrm{diam}\, |s'| \leq \frac{m}{m+1}\, \mathrm{diam}\, |s|$$

PROOF Let $\{p_j \mid 0 \le j \le m\}$ be points of $\mathbf{R}^n$ and assume that y is a convex combination of $\{p_j\}$ (that is, $y = \Sigma t_j p_j$, where $\Sigma t_j = 1$ and $t_j \ge 0$) and let $x \in \mathbf{R}^n$. Then

$$\|x - y\| \le \|x - \Sigma t_j p_j\| = \|\Sigma t_j(x - p_j)\| \le \Sigma t_j \|x - p_j\|$$

Therefore $\|x - y\| \le \sup \|x - p_j\|$. If x is also a convex combination of $\{p_j\}$, then $\|x - y\| \le \sup \|p_i - p_j\|$.

Regard $|s|$ as imbedded linearly in $\mathbf{R}^n$, with vertices $p_0, p_1, \ldots, p_m$. Then, by the above result, diam $|s| \le \sup \|p_i - p_j\|$, and if s' is a simplex of sd $\bar{s}$, diam $|s'| \le \sup \{\|p' - p''\| \mid p', p'' \in s'\}$. Therefore we need only show that if $p' = (p_0 + \cdots + p_q)/(q + 1)$ and $p'' = (p_0 + \cdots + p_r)/(r + 1)$, where $q \le r$, then $\|p' - p''\| \le [m/(m + 1)] \sup \|p_i - p_j\|$. Again by the result above,

$$\|p' - p''\| \le \sup \{\|p_i - p''\| \mid 0 \le i \le q\}$$

and also, for $0 \le i \le q$,

$$\|p_i - p''\| = \|p_i - \frac{1}{r+1} \sum_{0 \le j \le r} p_j\| \le \frac{1}{r+1} \sum_{0 \le j \le r} \|p_i - p_j\|$$

$$\le \frac{r}{r+1} \sup \|p_i - p_j\|$$

Therefore

$$\|p' - p''\| \le \frac{r}{r+1} \sup \{\|p_i - p_j\| \mid 0 \le i \le q, 0 \le j \le r\}$$

$$\le \frac{r}{r+1} \operatorname{diam} |s|$$

Because $r \le m$, $r/(r + 1) \le m/(m + 1)$ and diam $|s'| \le [m/(m + 1)]$ diam $|s|$. ∎

Given a metric on $|K|$, we define *mesh* of K by

$$\operatorname{mesh} K = \sup \{\operatorname{diam} |s| \mid s \in K\}$$

13 COROLLARY *If K is an m-dimensional complex and $|K|$ has a metric linear on K, then*

$$\operatorname{mesh} (\operatorname{sd} K) \le \frac{m}{m+1} \operatorname{mesh} K \quad ∎$$

This gives us the important result toward which we have been heading.

14 THEOREM *Let $\mathcal{U}$ be an open covering of a compact polyhedron X. Then X has triangulations finer than $\mathcal{U}$.*

PROOF Let (K,f) be a triangulation of X. We shall show that there exists an integer N such that if $n \ge N$, then $(\operatorname{sd}^n K, f)$ is finer than $\mathcal{U}$. Let $|K|$ be provided with a metric linear on K and let $\varepsilon > 0$ be a Lebesque number of the open covering $f^{-1}\mathcal{U} = \{f^{-1}U \mid U \in \mathcal{U}\}$ with respect to this metric [thus, if

$A \subset |K|$ and diam $A \le \varepsilon$, then $f(A)$ is contained in some element of $\mathfrak{U}$]. Such a number $\varepsilon > 0$ exists because $|K|$ is compact. Let $m = \dim K$ and choose N so that $[m/(m+1)]^N$ mesh $K \le \varepsilon/2$ (such an N exists because $\lim_{n\to\infty} [m/(m+1)]^n = 0$). If $n \ge N$, then, by corollary 13, mesh $\mathrm{sd}^n K \le \varepsilon/2$. If v' is any vertex of $\mathrm{sd}^n K$, diam (st v') ≤ 2 mesh $\mathrm{sd}^n K \le \varepsilon$. Therefore $f(\mathrm{st}\, v')$ is contained in some element of $\mathfrak{U}$, and $(\mathrm{sd}^n K, f)$ is finer than $\mathfrak{U}$ if $n \ge N$. ∎

This last result is true even if X is not compact. More precisely, if (K,f) is a triangulation of a polyhedron X and $\mathfrak{U}$ is an open covering of X, there exist subdivisions K' of K such that (K',f) is finer than $\mathfrak{U}$.[1] However, when X is not compact K' cannot generally be chosen to be an iterated barycentric subdivision of K, and so the proof for this case is more complicated than the proof of theorem 14. We need only the form proven in theorem 14, however, and so omit further consideration of the more general case.

4 SIMPLICIAL APPROXIMATION

A continuous map between the spaces of simplicial complexes can be suitably approximated by simplicial maps. This section contains a definition and characterization of the approximations and a proof of their existence for maps of a compact polyhedron into any polyhedron. Finally, we apply the result obtained to deduce some connectivity properties of spheres.

Let K_1 and K_2 be simplicial complexes and let $f: |K_1| \to |K_2|$ be continuous. A simplicial map $\varphi: K_1 \to K_2$ is called a *simplicial approximation* to f if $f(\alpha) \in \langle s_2 \rangle$ implies $|\varphi|(\alpha) \in |s_2|$ (or, equivalently, $f(\alpha) \in |s_2|$ implies $|\varphi|(\alpha) \in |s_2|$) for $\alpha \in |K_1|$ and $s_2 \in K_2$. Note that if v is a vertex of K_1 such that $f(v)$ is a vertex of K_2, then $|\varphi|(v) = f(v)$. Therefore we obtain the following result.

1 LEMMA *Let $f: |K_1| \to |K_2|$ be a map and suppose that for some subcomplex $L_1 \subset K_1$, $f \mid |L_1|$ is induced by a simplicial map $L_1 \to K_2$. If $\varphi: K_1 \to K_2$ is a simplicial approximation to f, then $|\varphi| \mid |L_1| = f \mid |L_1|$.* ∎

In particular, the only simplicial approximation to a map $|\varphi|: |K_1| \to |K_2|$ induced by a simplicial map $\varphi: K_1 \to K_2$ is φ itself. One sense in which a simplicial approximation is an approximation is the following.

2 LEMMA *Let $\varphi: K_1 \to K_2$ be a simplicial approximation to a map $f: |K_1| \to |K_2|$ and let $A \subset |K_1|$ be the subset of $|K_1|$ on which $|\varphi|$ and f agree. Then $|\varphi| \simeq f$ rel A.*

PROOF A homotopy relative to A from $|\varphi|$ to f is defined by the equation

$$F(\alpha,t) = tf(\alpha) + (1-t)(|\varphi|(\alpha)) \qquad \alpha \in |K_1|,\ t \in I$$

[1] See theorem 35 in J. H. C. Whitehead, Simplicial spaces, nuclei, and m-groups, *Proceedings of the London Mathematical Society*, vol. 45, pp. 243–327 (1939).

The right-hand side is well-defined, because if $f(\alpha) \in \langle s_2 \rangle$, then $|\varphi|(\alpha) \in |s_2|$, and so $F(\alpha,t) \in |s_2|$ for $t \in I$. The continuity of F is easily verified. Clearly, if $\alpha \in A$, then $F(\alpha,t) = f(\alpha)$ for all $t \in I$. Therefore F: $|\varphi| \simeq f$ rel A. ∎

The following theorem is a useful characterization of simplicial approximations.

3 THEOREM *A vertex map φ from K_1 to K_2 is a simplicial approximation to f: $|K_1| \to |K_2|$ if and only if for every vertex $v \in K_1$*

$$f(\text{st } v) \subset \text{st } \varphi(v)$$

PROOF Assume that φ is a simplicial approximation to f. Let $\alpha \in \text{st } v$ and suppose $f(\alpha) \in \langle s_2 \rangle$. Then $\alpha(v) \neq 0$ and $|\varphi|(\alpha) \in |s_2|$. Because φ is simplicial, $|\varphi|(\alpha)(\varphi(v)) \neq 0$. Therefore $\varphi(v)$ is a vertex of $|s_2|$, and $f(\alpha) \in \text{st } \varphi(v)$. Since this is so for every $\alpha \in \text{st } v$, $f(\text{st } v) \subset \text{st } \varphi(v)$.

Conversely, assume that φ is a vertex map such that $f(\text{st } v) \subset \text{st } \varphi(v)$ for every vertex $v \in K_1$. We show that φ is a simplicial map. If $\{v_i\}$ are vertices of a simplex of K_1, then $\bigcap \text{ st } v_i \neq \varnothing$ (by lemma 3.1.25) and

$$\varnothing \neq f(\bigcap \text{ st } v_i) \subset \bigcap f(\text{st } v_i) \subset \bigcap \text{ st } \varphi(v_i)$$

By lemma 3.1.25, $\{\varphi(v_i)\}$ are vertices of some simplex of K_2. Therefore φ is a simplicial map $K_1 \to K_2$.

To show that φ is a simplicial approximation to f, assume $\alpha \in \langle s_1 \rangle$ and $f(\alpha) \in \langle s_2 \rangle$ and let v be any vertex of s_1. Then $\alpha \in \text{ st } v$ and, by hypothesis, $f(\alpha) \in \text{st } \varphi(v)$. Therefore $\varphi(v)$ is a vertex of s_2. This is so for every vertex v of s_1. Because φ is simplicial, $|\varphi|(|s_1|) \subset |s_2|$. Hence $|\varphi|(\alpha) \in |s_2|$, and φ is a simplicial approximation to f. ∎

We are also interested in simplicial approximations φ: $(K_1,L_1) \to (K_2,L_2)$ to maps f: $(|K_1|,|L_1|) \to (|K_2|,|L_2|)$. The following corollary shows that any simplicial approximation $K_1 \to K_2$ to a map f: $(|K_1|,|L_1|) \to (|K_2|,|L_2|)$ is automatically a simplicial approximation when regarded as a map of pairs.

4 COROLLARY *Let f: $|K_1| \to |K_2|$ be a map such that $f(|L_1|) \subset |L_2|$ for $L_1 \subset K_1$ and $L_2 \subset K_2$ and let φ: $K_1 \to K_2$ be a simplicial approximation to f. Then $\varphi \mid L_1$ maps L_1 to L_2 and is a simplicial approximation to $f \mid |L_1|$.*

PROOF By theorem 3, it suffices to show that if v is a vertex of L_1, then $\varphi(v)$ is a vertex of L_2 such that

$$f(\text{st } v \cap |L_1|) \subset \text{st } \varphi(v) \cap |L_2|$$

Since φ is a simplicial approximation to f, $f(\text{st } v) \subset \text{st } \varphi(v)$, and if v is a vertex of L_1, then $f(v) \in \langle s_2 \rangle$ for some $s_2 \in L_2$ [because $f(|L_1|) \subset |L_2|$]. Therefore $\varphi(v)$ is a vertex of L_2 and

$$f(\text{st } v \cap |L_1|) \subset f(\text{st } v) \cap |L_2| \subset \text{st } \varphi(v) \cap |L_2| \quad ∎$$

It follows from corollary 4 that any simplicial approximation to a map

$f\colon (|K_1|,|L_1|) \to (|K_2|,|L_2|)$ is a simplicial map $\varphi\colon (K_1,L_1) \to (K_2,L_2)$. From lemma 2, it follows that $f \simeq |\varphi|$ as a map of pairs.

5 COROLLARY *The composite of simplicial approximations to maps is a simplicial approximation to the composite of the maps.*

PROOF Let $\varphi\colon K_1 \to K_2$ be a simplicial approximation to $f\colon |K_1| \to |K_2|$ and let $\psi\colon K_2 \to K_3$ be a simplicial approximation to $g\colon |K_2| \to |K_3|$. Then, by theorem 3, for a vertex $v \in K_1$

$$gf(\text{st } v) \subset g(\text{st } \varphi(v)) \subset \text{st } \psi\varphi(v)$$

and $\psi\varphi\colon K_1 \to K_3$ is thus a simplicial approximation to $gf\colon |K_1| \to |K_3|$. ∎

Theorem 3 leads to the following necessary and sufficient condition for the existence of a simplicial approximation to a map.

6 THEOREM *A map $f\colon |K_1| \to |K_2|$ admits simplicial approximations $K_1 \to K_2$ if and only if K_1 is finer than the open covering $\{f^{-1}(\text{st } v) \mid v$ is a vertex of $K_2\}$.*

PROOF By theorem 3, there exist simplicial approximations to f if and only if for each vertex $v_1 \in K_1$ there is a vertex $v_2 \in K_2$ such that $\text{st } v_1 \subset f^{-1}(\text{st } v_2)$. This is equivalent to the condition that K_1 is finer than $\{f^{-1}(\text{st } v)\}$ $v \in K_2$. ∎

If K' is a subdivision of K, then for vertices $v' \in K'$ and $v \in K$

$$v' \in \text{st}_K v \Leftrightarrow \text{st}_{K'} v' \subset \text{st}_K v$$

Combining this fact with theorem 3 yields the following corollary.

7 COROLLARY *Let K' be a subdivision of K. A vertex map φ from K' to K is a simplicial approximation to the identity map $|K'| \subset |K|$ if and only if $v' \in \text{st } \varphi(v')$ for every vertex $v' \in K'$.* ∎

In particular, if K' is a subdivision of K, there exist simplicial approximations $K' \to K$ to the identity map $|K'| \subset |K|$. Combining theorems 6 and 3.3.14 and corollary 4, we obtain the following *simplicial-approximation theorem.*

8 THEOREM *Let (K_1,L_1) be a finite simplicial pair and let $f\colon (|K_1|,|L_1|) \to (|K_2|,|L_2|)$ be a map. There exists an integer N such that if $n \geq N$ there are simplicial approximations $(\text{sd}^n K_1, \text{sd}^n L_1) \to (K_2,L_2)$ to f.* ∎

As remarked at the end of Sec. 3.3, theorem 3.3.14 is also valid for an arbitrary polyhedron X. Therefore, if K_1 is arbitrary and $f\colon |K_1| \to |K_2|$ is a map, there exists a subdivision K_1' of K_1 and a simplicial approximation $K_1' \to K_2$ to $f\colon |K_1'| \to |K_2|$. If K_1 is not finite, however, K_1' cannot generally be taken to be an iterated barycentric subdivision of K_1.

9 EXAMPLE If $\dot{s}$ is the complex consisting of all proper faces of a 2-simplex s, then $|\dot{s}|$ is homeomorphic to S^1, and therefore $[|\dot{s}|;|\dot{s}|]$ is an infinite set.

Because $\dot{s}$ is a finite complex, there are only a finite number of simplicial maps $\mathrm{sd}^n\, \dot{s} \to \dot{s}$ for any n. Therefore for any n there exist maps $|\dot{s}| \to |\dot{s}|$ having no simplicial approximation $\mathrm{sd}^n\, \dot{s} \to \dot{s}$.

10 EXAMPLE Let $\dot{s}$ be as in example 9 and let its vertices be v_0, v_1, v_2. Define $f\colon |\dot{s}| \to |\dot{s}|$ to be the map linear on sd $\dot{s}$ such that

$$f(v_0) = b\{v_0,v_1\} \qquad f(v_1) = b\{v_1,v_2\} \qquad f(v_2) = b\{v_2,v_0\}$$
$$f(b\{v_0,v_1\}) = v_1 \qquad f(b\{v_1,v_2\}) = v_2 \qquad f(b\{v_2,v_0\}) = v_0$$

Then $f \simeq |1_{\dot{s}}|$, but there is no simplicial approximation $\dot{s} \to \dot{s}$ to f. There are exactly eight simplicial approximations φ: sd $\dot{s} \to \dot{s}$ to f [φ is unique on $b\{v_0,v_1\}$, $b\{v_1,v_2\}$, and $b\{v_2,v_0\}$, and $\varphi(v_0) = v_0$ or v_1, $\varphi(v_1) = v_1$ or v_2, and $\varphi(v_2) = v_2$ or v_0].

As an application of the technique of simplicial approximation, we deduce the following useful result.

11 THEOREM *S^n is $(n-1)$-connected for $n \geq 1$.*

PROOF By theorem 1.6.7, it suffices to prove that if $m < n$, any map $S^m \to S^n$ is null homotopic. Let s_1 be an $(m+1)$-simplex and s_2 an $(n+1)$-simplex. Then S^m and S^n are homeomorphic, respectively, to $|\dot{s}_1|$ and $|\dot{s}_2|$. By theorem 8 and lemma 2, it suffices to show that if φ: $\mathrm{sd}^i\, \dot{s}_1 \to \dot{s}_2$ is any simplicial map, then $|\varphi|$ is null homotopic. Because dim $(\mathrm{sd}^i\, \dot{s}_1) = m < n$, φ maps $\mathrm{sd}^i\, \dot{s}_1$ into the m-dimensional skeleton of $\dot{s}_2$. Therefore there is some $\alpha \in |\dot{s}_2|$ such that

$$|\varphi|(|\mathrm{sd}^i\, \dot{s}_1|) \subset |\dot{s}_2| - \alpha$$

Because $|\dot{s}_2| - \alpha$ is homeomorphic to S^n minus a point, which is homeomorphic to $\mathbf{R}^n$, it is contractible. Therefore $|\varphi|$ is null homotopic. ∎

In particular, we have the following result.

12 COROLLARY *For $n > 1$, S^n is simply connected.* ∎

Because S^n is locally path connected, corollary 12 and the lifting theorem imply that any continuous map $f\colon S^n \to S^1$ can be factored through the covering map $ex\colon \mathbf{R} \to S^1$. Since $\mathbf{R}$ is contractible, this implies the following corollary.

13 COROLLARY *For $n > 1$ any continuous map $S^n \to S^1$ is null homotopic.* ∎

5 CONTIGUITY CLASSES

In the last section it was shown that any continuous map between the spaces of simplicial complexes has simplicial approximations defined on sufficiently fine subdivisions of the domain complex. In general, simplicial approximations to a given continuous map are not unique, and in this section we investigate this nonuniqueness.

We shall define an analogue of homotopy, called contiguity, in the category of simplicial pairs and simplicial maps. Different simplicial approximations to the same continuous map will be shown to the contiguous. The main result of the section is the existence of a bijection between the set of homotopy classes of continuous maps (from the space of a finite simplicial complex to the space of an arbitrary complex) and the direct limit of a certain sequence of contiguity classes of simplicial maps.

Let (K_1,L_1) and (K_2,L_2) be simplicial pairs. Two simplicial maps φ, φ': $(K_1,L_1) \to (K_2,L_2)$ are *contiguous* if, given a simplex $s \in K_1$ (or $s \in L_1$), $\varphi(s) \cup \varphi'(s)$ is a simplex of K_2 (or of L_2). Obviously, this is a reflexive and symmetric relation in the set of simplicial maps $(K_1,L_1) \to (K_2,L_2)$, but in general it is not transitive. There is, however, an equivalence relation, denoted by $\varphi \sim \varphi'$, in this set of simplicial maps that is defined by $\varphi \sim \varphi'$ if and only if there exists a finite sequence $\varphi_0, \varphi_1, \ldots, \varphi_n$ such that $\varphi_0 = \varphi$ and $\varphi_n = \varphi'$ and such that φ_{i-1} and φ_i are contiguous for $i = 1, 2, \ldots, n$. The corresponding equivalence classes are called *contiguity classes*, and the set of contiguity classes of simplicial maps from (K_1,L_1) to (K_2,L_2) is denoted by $[K_1,L_1; K_2,L_2]$. If φ: $(K_1,L_1) \to (K_2,L_2)$ is a simplicial map, its contiguity class is denoted by $[\varphi]$.

We shall see that contiguity classes are algebraic analogues of homotopy classes. We begin by showing that contiguity classes can be composed.

1 LEMMA *Composites of contiguous simplicial maps are contiguous.*

PROOF Assume that φ, φ': $(K_1,L_1) \to (K_2,L_2)$ are contiguous and ψ, ψ': $(K_2,L_2) \to (K_3,L_3)$ are contiguous. If s is a simplex of K_1 (or L_1), $\varphi(s) \cup \varphi'(s)$ is a simplex of K_2 (or L_2). Therefore

$$\psi(\varphi(s) \cup \varphi'(s)) \cup \psi'(\varphi(s) \cup \varphi'(s))$$

is a simplex of K_3 (or L_3). This implies that the subset $\psi\varphi(s) \cup \psi'\varphi'(s)$ is a simplex of K_3 (or L_3) and that $\psi\varphi$, $\psi'\varphi'$: $(K_1,L_1) \to (K_3,L_3)$ are contiguous. ∎

It follows easily from lemma 1 that if $\varphi \sim \varphi'$ and $\psi \sim \psi'$, then $\psi\varphi \sim \psi\varphi' \sim \psi'\varphi'$. Therefore there is a well-defined composite of contiguity classes

$$[\psi] \circ [\varphi] = [\psi\varphi]$$

for $(K_1,L_1) \xrightarrow{\varphi} (K_2,L_2) \xrightarrow{\psi} (K_3,L_3)$. Thus there is a *contiguity category* whose objects are simplicial pairs and whose morphisms are contiguity classes of simplicial pairs. There are full subcategories of the contiguity category determined by the pairs $(K,\varnothing)$ or by the pointed simplicial complexes.

2 LEMMA *Contiguous simplicial maps which agree on a subcomplex define continuous maps which are homotopic relative to the space of the subcomplex.*

PROOF Assume that φ, φ': $(K_1,L_1) \to (K_2,L_2)$ are contiguous and agree on $L \subset K_1$. Define a homotopy F: $(|K_1| \times I, |L_1| \times I) \to (|K_2|,|L_2|)$ rel $|L|$ from $|\varphi|$ to $|\varphi'|$ by the equation

$$F(\alpha,t) = (1 - t)(|\varphi|(\alpha)) + t(|\varphi'|(\alpha)) \qquad \alpha \in |K_1|,\ t \in I \quad \blacksquare$$

Since homotopy is an equivalence relation, if $\varphi \sim \varphi'$, then $|\varphi| \simeq |\varphi'|$. Therefore we have the following result.

3 COROLLARY *There is a covariant functor from the contiguity category of simplicial pairs to the homotopy category of topological pairs which assigns to (K,L) the pair $(|K|,|L|)$ and to $[\varphi]$ the homotopy class $[|\varphi|]$.* ■

The next result considers different simplicial approximations to the same continuous map.

4 LEMMA *Two simplicial approximations $(K_1,L_1) \to (K_2,L_2)$ to the same continuous map are contiguous.*

PROOF Let φ, φ': $(K_1,L_1) \to (K_2,L_2)$ be simplicial approximations to f: $(|K_1|,|L_1|) \to (|K_2|,|L_2|)$ and let $\{v_i\}$ be a simplex of K_1. Then $\bigcap$ st $v_i \neq \varnothing$, and by theorem 3.4.3,

$$\varnothing \neq f(\bigcap \text{st } v_i) \subset \bigcap f(\text{st } v_i) \subset \bigcap (\text{st } \varphi(v_i) \cap \text{st } \varphi'(v_i))$$

Therefore $\{\varphi(v_i)\} \cup \{\varphi'(v_i)\}$ is a simplex of K_2. If $\{v_i\}$ is a simplex of L_1, a similar argument shows that $\{\varphi(v_i)\} \cup \{\varphi'(v_i)\}$ is a simplex of L_2. Therefore φ and φ' are contiguous. ■

Since it was necessary to subdivide in order to obtain simplicial approximations to arbitrary continuous maps, we should also expect to subdivide to make contiguity classes correspond to homotopy classes. An example will illustrate the relation between homotopy and contiguity.

5 EXAMPLE Let s be a 2-simplex with vertices v_0, v_1, v_2 and let $K_1 = K_2 = \dot{s}$. Any vertex map from K_1 to K_2 is a simplicial map. Therefore there are exactly 27 simplicial maps $K_1 \to K_2$. Of these 27, there are 21 which map K_1 into a proper subcomplex of K_2, and these constitute one contiguity class. Of the remaining 6, each is the only element of its contiguity class, the 3 even permutations of the vertices defining homotopic continuous maps corresponding to one generator of the group

$$[|K_1|;|K_2|] \approx [S^1;S^1] \approx \mathbf{Z}$$

and the 3 odd permutations corresponding to the other generator of this group. Therefore $[K_1;K_2]$ consists of 7 elements, and the image

$$[K_1;K_2] \to [|K_1|;|K_2|]$$

consists of 3 elements.

This example shows that simplicial maps which define homotopic continuous maps need not be in the same contiguity class. The next result shows that a finite simplicial complex can be subdivided so that homotopic simplicial maps from it to some other complex can be simplicially approximated on the subdivision by maps in the same contiguity class; it is the analogue for homotopy of the simplicial-approximation theorem.

6 THEOREM *Let $f, f'\colon (|K_1|,|L_1|) \to (|K_2|,|L_2|)$ be homotopic, where K_1 is finite. Then there exists N such that f and f' have simplicial approximations*

$$\varphi, \varphi'\colon (\mathrm{sd}^N K_1, \mathrm{sd}^N L_1) \to (K_2,L_2)$$

respectively, in the same contiguity class.

PROOF Let $F\colon (|K_1| \times I, |L_1| \times I) \to (|K_2|,|L_2|)$ be a homotopy from f to f'. Because $|K_1|$ is compact, there exists a sequence $0 = t_0 < t_1 < \cdots < t_n = 1$ of points of I such that for $\alpha \in |K_1|$ and $i = 1, 2, \ldots, n$ there is a vertex $v \in K_2$ such that $F(\alpha,t_{i-1})$ and $F(\alpha,t_i)$ both belong to st v. Let $f_i\colon (|K_1|,|L_1|) \to (|K_2|,|L_2|)$ be defined by $f_i(\alpha) = F(\alpha,t_i)$. Then $f = f_0$ and $f' = f_n$, and for $i = 1,2, \ldots, n$ the set

$$\mathcal{U}_i = \{f_i^{-1}(\mathrm{st}\ v) \cap f_{i-1}^{-1}(\mathrm{st}\ v) \mid v \in K_2\}$$

is an open covering of $|K_1|$. Let N be chosen large enough so that $\mathrm{sd}^N K_1$ is finer than $\mathcal{U}_1, \mathcal{U}_2, \ldots, \mathcal{U}_n$ (which is possible, by theorem 3.3.14). For $i = 1, 2, \ldots, n$ let φ_i be a vertex map from $\mathrm{sd}^N K_1$ to K_2 such that

$$f_i(\mathrm{st}\ v) \cup f_{i-1}(\mathrm{st}\ v) \subset \mathrm{st}\ \varphi_i(v)$$

for each vertex $v \in K_1$ (such a vertex map φ_i exists because $\mathrm{sd}^N K_1$ is finer than $\mathcal{U}_i$). By theorem 3.4.3,

$$\varphi_i\colon (\mathrm{sd}^N K_1, \mathrm{sd}^N L_1) \to (K_2,L_2)$$

is a simplicial approximation to f_i and to f_{i-1}. Because φ_i and φ_{i+1} are simplicial approximations to f_i, it follows from lemma 4 that φ_i and φ_{i+1} are contiguous for $i = 1, 2, \ldots, n-1$. Therefore $\varphi_1 \sim \varphi_n$, and also φ_1 is a simplicial approximation to $f_0 = f$ and φ_n is a simplicial approximation to $f_n = f'$. ∎

Unlike the simplicial-approximation theorem, this last result is definitely false if K_1 is not a finite simplicial complex. That is, given homotopic maps $f, f'\colon |K_1| \to |K_2|$, there need not be a subdivision K_1' of K_1 such that f and f' have simplicial approximations $K_1' \to K_2$ in the same contiguity class.

7 EXAMPLE Let $K_1 = K_2$ equal the simplicial complex of example 3.1.8, with space homeomorphic to $\mathbf{R}$. Let $\varphi\colon K_1 \to K_2$ be the identity simplicial map and $\varphi'\colon K_1 \to K_2$ be the constant simplicial map sending every vertex of K_1 to the vertex 0 of K_2. Since $\mathbf{R}$ is contractible, $|\varphi| \simeq |\varphi'|$. However, if K_1' is any subdivision of K_1, a simplicial approximation $\psi\colon K_1' \to K_2$ to $|\varphi|$ must be surjective to the vertices of K_2 and a simplicial approximation $\psi'\colon K_1' \to K_2$ to $|\varphi'|$ must be the constant map to 0. Since two contiguous maps $K_1' \to K_2$ either both map onto a finite set of vertices or neither does, ψ and ψ' are not in the same contiguity class.

We show that if K_1 is finite the set of homotopy classes of maps $[|K_1|,|L_1|;\ |K_2|,|L_2|]$ is the direct limit of the set of contiguity classes

$$[\mathrm{sd}^n\ K_1,\ \mathrm{sd}^n\ L_1;\ K_2,L_2]$$

Note that simplicial approximations $(\mathrm{sd}\ K_1,\ \mathrm{sd}\ L_1) \to (K_1,L_1)$ to the identity map $(|\mathrm{sd}\ K_1|,\ |\mathrm{sd}\ L_1|) \subset (|K_1|,|L_1|)$ exist, by corollary 3.4.7, and any two are contiguous, by lemma 4. Because the composites of contiguous simplicial maps are contiguous by lemma 1, there is a well-defined map

$$sd\colon [K_1,L_1;\ K_2,L_2] \to [\mathrm{sd}\ K_1,\ \mathrm{sd}\ L_1;\ K_2,L_2]$$

defined by

$$sd[\varphi] = [\varphi\lambda]$$

where $\lambda\colon (\mathrm{sd}\ K_1,\ \mathrm{sd}\ L_1) \to (K_1,L_1)$ is any simplicial approximation to the identity $(|\mathrm{sd}\ K_1|,\ |\mathrm{sd}\ L_1|) \subset (|K_1|,|L_1|)$ and $\varphi\colon (K_1,L_1) \to (K_2,L_2)$ is an arbitrary simplicial map. By iteration there is obtained a sequence

$$\cdots \to [\mathrm{sd}^n\ K_1,\ \mathrm{sd}^n\ L_1;\ K_2,L_2] \xrightarrow{sd} [\mathrm{sd}^{n+1}\ K_1,\ \mathrm{sd}^{n+1}\ L_1;\ K_2,L_2] \to \cdots$$

which begins with $[K_1,L_1;\ K_2,L_2]$ on the left and extends indefinitely on the right. The direct limit $\lim_{\to} \{[\mathrm{sd}^n\ K_1,\ \mathrm{sd}^n\ L_1;\ K_2,L_2]\}$ is a functor of two arguments contravariant in (K_1,L_1) and covariant in (K_2,L_2). For finite K_1 this functor is naturally equivalent to the functor $[|K_1|,|L_1|;\ |K_2|,|L_2|]$.

8 THEOREM *If K_1 is a finite simplicial complex, there is a natural equivalence*

$$\lim_{\to} \{[\mathrm{sd}^n\ K_1,\ \mathrm{sd}^n\ L_1;\ K_2,L_2]\} \approx [|K_1|,|L_1|;\ |K_2|,|L_2|]$$

PROOF A function from the direct limit to $[|K_1|,|L_1|;\ |K_2|,|L_2|]$ consists of a sequence of functions

$$f_n\colon [\mathrm{sd}^n\ K_1,\ \mathrm{sd}^n\ L_1;\ K_2,L_2] \to [|K_1|,|L_1|;\ |K_2|,|L_2|]$$

for $n \geq 0$ such that $f_n = f_{n+1} \circ sd$ for $n \geq 0$. Such a sequence f_n is defined by $f_n[\varphi] = [|\varphi|]$ for $\varphi\colon (\mathrm{sd}^n\ K_1,\ \mathrm{sd}^n\ L_1) \to (K_2,L_2)$, because if

$$\lambda_n\colon (\mathrm{sd}^{n+1}\ K_1,\ \mathrm{sd}^{n+1}\ L_1) \to (\mathrm{sd}^n\ K_1,\ \mathrm{sd}^n\ L_1)$$

is a simplicial approximation to the identity map

$$(|\mathrm{sd}^{n+1}K_1|,\ |\mathrm{sd}^{n+1}L_1|) \subset (|\mathrm{sd}^nK_1|,\ |\mathrm{sd}^nL_1|)$$

then, by lemma 3.4.2, $|\lambda_n| \simeq 1$, and

$$f_{n+1}\ sd[\varphi] = [|\varphi\lambda_n|] = [|\varphi|] = f_n[\varphi]$$

The sequence $\{f_n\}$ defines a natural transformation

$$f\colon \lim_{\to} \{[\mathrm{sd}^n\ K_1,\ \mathrm{sd}^n\ L_1;\ K_2,L_2]\} \to [|K_1|,|L_1|;\ |K_2|,|L_2|]$$

and we show that f is a bijection.

It follows easily from the simplicial-approximation theorem that $\{f_n\}$ satisfies (*a*) of theorem 1.3 of the Introduction; for if $g\colon (|K_1|,|L_1|) \to (|K_2|,|L_2|)$

is a map and φ: $(\mathrm{sd}^n K_1, \mathrm{sd}^n L_1) \to (K_2,L_2)$ is a simplicial approximation to g, then $|\varphi| \simeq g$, and

$$f_n[\varphi] = [|\varphi|] = [g]$$

To show that $\{f_n\}$ satisfies (*b*) of theorem 1.3 of the Introduction, assume

$$\varphi, \varphi': (\mathrm{sd}^n K_1, \mathrm{sd}^n \mathrm{L}_1) \to (K_2,L_2)$$

are such that $|\varphi| \simeq |\varphi'|$. By theorem 6, there exists $m \geq n$ such that $|\varphi|$ and $|\varphi'|$ have simplicial approximations

$$\psi, \psi': (\mathrm{sd}^m K_1, \mathrm{sd}^m L_1) \to (K_2,L_2)$$

in the same contiguity class. Let

$$\lambda_{m,n}: (\mathrm{sd}^m K_1, \mathrm{sd}^m L_1) \to (\mathrm{sd}^n K_1, \mathrm{sd}^n L_1)$$

be the composite $\lambda_{m,n} = \lambda_n \lambda_{n+1} \cdots \lambda_{m-1}$. Then $\lambda_{m,n}$ is a simplicial approximation to the identity map, and because φ is a simplicial approximation to $|\varphi|$, $\varphi\lambda_{m,n}$ is also a simplicial approximation to $|\varphi|$, by corollary 3.4.5. By lemma 4, $\varphi\lambda_{m,n}$ is contiguous to ψ. Similarly, $\varphi'\lambda_{m,n}$ is contiguous to ψ'. Since ψ and ψ' are in the same contiguity class, so are $\varphi\lambda_{m,n}$ and $\varphi'\lambda_{m,n}$. This means that $sd^{m-n}[\varphi] = sd^{m-n}[\varphi']$ in $[\mathrm{sd}^m K_1, \mathrm{sd}^m L_1; K_2,L_2]$. ∎

For finite K_1 the last result furnishes an algebraic description of the set $[|K_1|,|L_1|; |K_2|,|L_2|]$. As an application, note that if K_2 is a countable complex, there are only a countable number of simplicial maps $(\mathrm{sd}^n K_1, \mathrm{sd}^n L_1) \to (K_2,L_2)$ for $n \geq 0$. Therefore $[\mathrm{sd}^n K_1, \mathrm{sd}^n L_1; K_2,L_2]$ is countable for $n \geq 0$. Because the direct limit of a sequence of countable sets is countable, we obtain the following result.

9 COROLLARY *Let (X,A) and (Y,B) be polyhedral pairs with X compact and Y the space of a countable complex. Then $[X,A; Y,B]$ is a countable set.* ∎

6 THE EDGE-PATH GROUPOID

It was shown in the last section that for finite K_1, $[|K_1|;|K_2|]$ is describable as a limit in which K_1 is subdivided but K_2 is not. In particular, for any simplicial complex K the set of path classes of $|K|$ from v_0 to v_1 is determined by the simplicial structure of K. This is made explicit in the present section, where we define a simplicial analogue of the fundamental groupoid of a space. In the next section the fundamental group of a polyhedron is presented in terms of generators and relations.

An *edge* of a simplicial complex K is an ordered pair of vertices (v,v') which belong to some simplex of K. The first vertex v is called the *origin* of the edge, and the second vertex v' is called the *end* of the edge. An *edge path* ζ of K is a finite nonempty sequence $e_1e_2 \cdots e_r$ of edges of K such that end

$e_i = \text{orig}\, e_{i+1}$ for $i = 1, \ldots, r-1$. We define $\text{orig}\, \zeta = \text{orig}\, e_1$ and $\text{end}\, \zeta = \text{end}\, e_r$. A *closed edge path* at $v_0 \in K$ is an edge path ζ such that $\text{orig}\, \zeta = v_0 = \text{end}\, \zeta$. If ζ_1 and ζ_2 are edge paths of K such that $\text{end}\, \zeta_1 = \text{orig}\, \zeta_2$, we define the *product edge path* $\zeta_1\zeta_2$ to be the edge path consisting of the sequence of edges of ζ_1 followed by the sequence of edges of ζ_2. Then $\text{orig}\, \zeta_1\zeta_2 = \text{orig}\, \zeta_1$ and $\text{end}\, \zeta_1\zeta_2 = \text{end}\, \zeta_2$. It is clear that if $\text{end}\, \zeta_1 = \text{orig}\, \zeta_2$ and $\text{end}\, \zeta_2 = \text{orig}\, \zeta_3$, then $\zeta_1(\zeta_2\zeta_3) = (\zeta_1\zeta_2)\zeta_3$. The product of edge paths thus satisfies associativity; however, there are no left or right identity elements for the product. To obtain a category (as was done for paths in a topological space) it is necessary to define an equivalence relation in the set of edge paths of K.

Two edge paths ζ and ζ' in K are *simply equivalent* if there exist vertices v, v', and v'' of K belonging to some simplex of K such that the unordered pair $\{\zeta,\zeta'\}$ equals one of the following:

The unordered pair $\{(v,v''), (v,v')(v',v'')\}$
The unordered pair $\{\zeta_1(v,v''), \zeta_1(v,v')(v',v'')\}$ for some edge path ζ_1 in K with $\text{end}\, \zeta_1 = v$
The unordered pair $\{(v,v'')\zeta_2, (v,v')(v',v'')\zeta_2\}$ for some edge path ζ_2 in K with $\text{orig}\, \zeta_2 = v''$
The unordered pair $\{\zeta_1(v,v'')\zeta_2, \zeta_1(v,v')(v',v'')\zeta_2\}$ for some edge paths ζ_1 and ζ_2 in K with $\text{end}\, \zeta_1 = v$ and $\text{orig}\, \zeta_2 = v''$.

Two edge paths ζ and ζ' will be said to be *equivalent*, denoted by $\zeta \sim \zeta'$, if there is a finite sequence of edge paths $\zeta_0, \zeta_1, \ldots, \zeta_n$ such that $\zeta = \zeta_0$ and $\zeta' = \zeta_n$, and ζ_{i-1} and ζ_i are simply equivalent for $i = 1, \ldots, n$. This is an equivalence relation, and the following statements are easily verified.

1 *$\zeta \sim \zeta'$ implies that ζ and ζ' have the same origin and the same end.* ∎

2 *$\zeta_1 \sim \zeta_1'$, $\zeta_2 \sim \zeta_2'$ and* $\text{end}\, \zeta_1 = \text{orig}\, \zeta_2$ *imply $\zeta_1\zeta_2 \sim \zeta_1'\zeta_2'$.* ∎

3 *If* $\text{orig}\, \zeta = v_1$ *and* $\text{end}\, \zeta = v_2$, *then $(v_1,v_1)\zeta \sim \zeta \sim \zeta(v_2,v_2)$.* ∎

If ζ is an edge path, $[\zeta]$ denotes its equivalence class. It follows from statement 1 that $\text{orig}\, [\zeta]$ and $\text{end}\, [\zeta]$ are well-defined (by $\text{orig}\, [\zeta] = \text{orig}\, \zeta$ and $\text{end}\, [\zeta] = \text{end}\, \zeta$). From statement 2 there is a well-defined composite $[\zeta_1] \circ [\zeta_2] = [\zeta_1\zeta_2]$ defined if $\text{end}\, \zeta_1 = \text{orig}\, \zeta_2$. We then have the following simplicial analogue of theorem 1.7.7.

4 THEOREM *There is a category $\mathcal{E}(K)$ whose objects are the vertices of K and whose morphisms from v_1 to v_0 are the equivalence classes $[\zeta]$ with* $\text{orig}\, [\zeta] = v_0$ *and* $\text{end}\, [\zeta] = v_1$ *and whose composite is $[\zeta_1] \circ [\zeta_2]$.*

PROOF The existence of identity morphisms follows from statement 3, and the associativity of the composite is a consequence of the associativity of the product of edge paths. ∎

We now show that $\mathcal{E}(K)$ is a groupoid. If $e = (v,v')$ is an edge of K, we define $e^{-1} = (v',v)$, and if $\zeta = e_1 \cdots e_r$ is an edge path in K, we define $\zeta^{-1} = e_r^{-1} \cdots e_1^{-1}$. The following statements are then easily verified.

5 $(\zeta^{-1})^{-1} = \zeta$. ■

6 orig ζ^{-1} = end ζ *and* end ζ^{-1} = orig ζ. ■

7 $\zeta_1 \sim \zeta_2$ *implies* $\zeta_1^{-1} \sim \zeta_2^{-1}$. ■

8 *If* orig $\zeta = v_1$ *and* end $\zeta = v_2$, *then* $\zeta\zeta^{-1} \sim (v_1,v_1)$ *and* $\zeta^{-1}\zeta \sim (v_2,v_2)$. ■

It follows that in $\mathcal{E}(K)$, $[\zeta^{-1}] = [\zeta]^{-1}$, and so $\mathcal{E}(K)$ is a groupoid. This groupoid is called the *edge-path groupoid* of K. If v_0 is a vertex of K, the operation $[\zeta] \circ [\zeta']$ in the set of elements of $\mathcal{E}(K)$ with origin and end at v_0 gives a group denoted by $E(K,v_0)$ and is called the *edge-path group of K with base vertex* v_0.

To compare $\mathcal{E}(K)$ [and $E(K,v_0)$] with $\mathcal{P}(|K|)$ [and $\pi(|K|,v_0)$], for $r \geq 1$ let I_r denote the subdivision of I into r equal subintervals; that is, I_r is the simplicial complex

$$I_r = \left\{\left\{\frac{i}{r}\right\}\middle|\, 0 \leq i \leq r\right\} \cup \left\{\left\{\frac{i-1}{r}, \frac{i}{r}\right\}\middle|\, 1 \leq i \leq r\right\}$$

Given an edge path $\zeta = e_1 \cdots e_r$ in K with r edges, let $\varphi_\zeta\colon I_r \to K$ be the simplicial map defined by

$$\varphi_\zeta\left(\frac{i}{r}\right) = \begin{cases} \text{orig } e_{i+1} & 0 \leq i \leq r-1 \\ \text{end } e_i & 1 \leq i \leq r \end{cases}$$

Then $|\varphi_\zeta|\colon I \to |K|$ is a path in $|K|$, and it is easily seen that the following statements hold.

9 orig $|\varphi_\zeta|$ = orig ζ *and* end $|\varphi_\zeta|$ = end ζ. ■

10 $\zeta \sim \zeta'$ *implies* $|\varphi_\zeta| \simeq |\varphi_{\zeta'}|$ rel $\dot{I}$. ■

11 *If* end ζ_1 = orig ζ_2, *then* $|\varphi_{\zeta_1\zeta_2}| \simeq |\varphi_{\zeta_1}| * |\varphi_{\zeta_2}|$ rel $\dot{I}$. ■

It follows that there is a natural transformation ρ from $\mathcal{E}(K)$ to $\mathcal{P}(|K|)$ which assigns to $v \in K$ the point $v \in |K|$ and to a morphism $[\zeta]$ in $\mathcal{E}(K)$ the morphism $[|\varphi_\zeta|]$ in $\mathcal{P}(|K|)$. We shall prove that for vertices $v_0, v_1 \in K$, ρ is a bijection

$$\rho\colon \hom_{\mathcal{E}}(v_1,v_0) \approx \hom_{\mathcal{P}}(v_1,v_0)$$

This can be obtained from theorem 3.5.8, but there is also a direct proof (which seems no longer than a proof based on theorem 3.5.8).

12 LEMMA *For any* $v_0, v_1 \in K$ *the function*

$$\rho\colon \hom_{\mathcal{E}}(v_1,v_0) \to \hom_{\mathcal{P}}(v_1,v_0)$$

is surjective.

PROOF Given a path $\omega\colon I \to |K|$ from v_0 to v_1, because $I = |I_1|$, it follows from theorem 3.4.8 that there is a simplicial map

$$\varphi\colon \mathrm{sd}^n I_1 \to K$$

which is a simplicial approximation to ω. Since $\mathrm{sd}^n I_1 = I_{2^n}$, there is an edge path $\zeta = e_1 \cdots e_{2^n}$ defined by $e_i = (\varphi((i-1)/2^n), \varphi(i/2^n))$ such that $\varphi = \varphi_\zeta$ for this ζ. Because $\varphi(0) = \omega(0)$ and $\varphi(1) = \omega(1)$, it follows from lemma 3.4.2 that $|\varphi| \simeq \omega$ rel $\dot{I}$. Therefore $[\omega] = [|\varphi|] = [|\varphi_\zeta|] = \rho[\zeta]$. ∎

We shall need some preliminary lemmas before showing that ρ is injective.

13 LEMMA *For any simplex s two edge paths in $\bar{s}$ with the same origin and the same end are equivalent.*

PROOF It suffices to prove that if ζ is any edge path in $\bar{s}$ from orig $\zeta = v$ and end $\zeta = v'$, then ζ is equivalent to the edge path consisting of the single edge (v,v'). This is proved by induction on the number of edges of ζ. ∎

14 LEMMA *Let ζ and ζ' be edge paths in K, each with r edges, such that $\varphi_\zeta, \varphi_{\zeta'}\colon I_r \to K$ are contiguous. Then $\zeta \sim \zeta'$.*

PROOF Let $\zeta = e_1 \cdots e_r$, where $e_i = (v_{i-1},v_i)$, and let $\zeta' = e'_1 \cdots e'_r$, where $e'_i = (v'_{i-1},v'_i)$. For $0 \le i \le r$ let $\bar{e}_i = (v_i,v'_i)$ (this is an edge of K because φ_ζ and $\varphi_{\zeta'}$ are contiguous). From the definition of equivalence

$$\zeta \sim e_1\bar{e}_1\bar{e}_1{}^{-1}e_2\bar{e}_2 \cdots \bar{e}_{r-1}^{-1}e_r$$

Because φ_ζ and $\varphi_{\zeta'}$ are contiguous, for each $1 \le i \le r$ there is some simplex s_i of K such that e_i, e'_i, $\bar{e}_{i-1}$, and $\bar{e}_i$ all are edges of $\bar{s}_i$. By lemma 13, $e_1\bar{e}_1 \sim e'_1$ and $\bar{e}_{i-1}^{-1}e_i\bar{e}_i \sim e'_i$ for $2 \le i \le r-1$, and $\bar{e}_{r-1}^{-1}e_r \sim e'_r$. Therefore

$$e_1\bar{e}_1\bar{e}_1^{-1}e_2\bar{e}_2 \cdots \bar{e}_{r-1}^{-1}e_r \sim e'_1e'_2 \cdots e'_r = \zeta' \quad ∎$$

15 LEMMA *Let $\zeta = e_1 \cdots e_r$ be an edge path of K and let $\lambda\colon I_{2r} \to I_r$ be a simplicial approximation to the identity map $|I_{2r}| \subset |I_r|$. Then $\varphi_\zeta\lambda = \varphi_{\zeta'}\colon I_{2r} \to K$ for some $\zeta' = e'_1 \cdots e'_{2r}$ and $\zeta \sim \zeta'$.*

PROOF Let $e_i = (v_{i-1},v_i)$ for $0 \le i \le r$. Then $e'_{2i-1}e'_{2i} = (v_{i-1},\bar{v}_i)(\bar{v}_i.v_i)$ for a vertex $\bar{v}_i$ which equals either v_{i-1} or v_i. By lemma 13, $e'_{2i-1}e'_{2i} \sim e_i$ and $\zeta' \sim \zeta$. ∎

We are now ready for the main result on the edge-path groupoid.

16 THEOREM *For vertices v_0, $v_1 \in K$ the function*

$$\rho\colon \hom_{\mathfrak{E}}(v_1,v_0) \to \hom_{\mathcal{P}}(v_1,v_0)$$

is a bijection.

PROOF In view of lemma 12, it only remains to prove that ρ is injective. Assume that ζ and ζ' are edge paths from v_0 to v_1 such that $|\varphi_\zeta| \simeq |\varphi_{\zeta'}|$ rel $\dot{I}$. By juxtaposing trivial edges (v_1,v_1) at the end of ζ or ζ' sufficiently often, we can replace ζ and ζ' by equivalent edge paths having an equal number of edges. Hence there is no loss of generality in assuming ζ and ζ' both to have r edges. Then $\varphi_\zeta, \varphi_{\zeta'}: I_r \to K$ are such that $|\varphi_\zeta| \simeq |\varphi_{\zeta'}|$ rel $\dot{I}$. By theorem 3.5.6, there exists m such that if λ: $\text{sd}^m I_r \to I_r$ is a simplicial approximation to the identity, then $\varphi_\zeta\lambda$ and $\varphi_{\zeta'}\lambda$ are in the same contiguity class. Now $\varphi_\zeta\lambda = \varphi_{\zeta_1}$ and $\varphi_{\zeta'}\lambda = \varphi_{\zeta_1'}$ for edge paths ζ_1 and ζ_1' in K. By lemma 15 (and induction on m), $\zeta \sim \zeta_1$ and $\zeta' \sim \zeta_1'$. From lemma 14 it follows that $\zeta_1 \sim \zeta_1'$. Therefore $\zeta \sim \zeta'$. ∎

If φ: $K_1 \to K_2$ is a simplicial map, there is a covariant functor $\varphi_\#$: $\mathcal{E}(K_1) \to \mathcal{E}(K_2)$ defined by

$$\varphi_\#[\zeta] = [\varphi(\zeta)]$$

where, if $\zeta = (v_0,v_1)(v_1,v_2) \cdots (v_{r-1},v_r)$, then $\varphi(\zeta) = (\varphi(v_0),\varphi(v_1)) \cdots (\varphi(v_{r-1}),\varphi(v_r))$. It is trivial to verify that commutativity holds in the square

$$\begin{array}{ccc} \mathcal{E}(K_1) & \xrightarrow{\varphi_\#} & \mathcal{E}(K_2) \\ \rho\downarrow & & \downarrow\rho \\ \mathcal{P}(|K_1|) & \xrightarrow{|\varphi|_\#} & \mathcal{P}(|K_2|) \end{array}$$

Therefore we have the following result.

17 COROLLARY *On the category of pointed simplicial complexes K with base vertex v_0, ρ is a natural equivalence of the covariant functor $E(K,v_0)$ with the covariant functor $\pi(|K|,v_0)$.* ∎

Note that $\mathcal{E}(K)$ is determined by the 2-skeleton of K; that is, the edge paths of K are determined by pairs of vertices of K which are vertices of a simplex, and the equivalences between edge paths are determined by triples of vertices which are vertices of a simplex. Hence $\mathcal{E}(K^2) \approx \mathcal{E}(K)$.

18 COROLLARY *For any pointed simplicial complex (K,v_0), the inclusion map $K^2 \subset K$ induces an isomorphism*

$$\pi(|K^2|,v_0) \approx \pi(|K|,v_0) \quad ∎$$

If s is a simplex of K, any two of its vertices belong to the same component of $\mathcal{E}(K)$. Therefore the components $\{E_j\}$ of $\mathcal{E}(K)$ define a partition of K into subcomplexes $\{K_j\}$, called the *components* of K, defined by $K_j = \{s \in K \mid s$ has some vertex in $E_j\}$. K is said to be *connected* if it contains exactly one component.

19 THEOREM *If $\{K_j\}$ are the components of K, then $\{|K_j|\}$ are the path components of $|K|$.*

PROOF If v is a vertex of K, then st v is path connected and so belongs to the same path component of $|K|$ as v. It follows from theorem 16 that two vertices of K are in the same component of $\mathcal{P}(|K|)$ if and only if they are in the same component of $\mathcal{E}(K)$. Therefore, if $\{E_j\}$ is the set of components of $\mathcal{E}(K)$, the path components of $|K|$ are the sets $\{P_j\}$ defined by $P_j = \cup\{\text{st } v \mid v \in E_j\}$. Clearly, $P_j = |K_j|$. ∎

7 GRAPHS

We show how a system of generators and relations for the edge-path group $E(K,v_0)$ can be determined. This provides a method for finding generators and relations of the fundamental group of a polyhedron. Since every edge path of K is an edge path of the one-dimensional skeleton of K, we start with a description of the edge path group of a one-dimensional complex.

A one-dimensional simplicial complex is called a *graph*. A *tree* is defined to be a simply connected graph.

1 LEMMA *A graph is a tree if and only if it is contractible.*

PROOF Since a contractible space is simply connected, a contractible graph is a tree. Conversely, let K be a tree and let α_0 be a point of $|K|$. We shall define a homotopy $F\colon |K| \times I \to |K|$ from the identity map 1 of $|K|$ to the constant map c of $|K|$ to α_0. Since $|K|$ is path connected, for each vertex v of K there is a path ω_v in $|K|$ from v to α_0. We now define F on $v \times I$ by $F(v,t) = \omega_v(t)$. For every 1-simplex s of K the map F is defined on the subset $(|s| \times 0) \cup (|s| \times 1) \cup (|\dot{s}| \times I)$ of $|s| \times I$. Since $|K|$ is simply connected and $(|s| \times I, (|s| \times 0) \cup (|s| \times 1) \cup (|\dot{s}| \times I))$ is homeomorphic to (E^2,S^1), it follows that F can be extended over $|s| \times I$. In this way we obtain a map $F\colon |K| \times I \to |K|$ whose restriction to each $|s| \times I$ is continuous. Then F is continuous and $F\colon 1 \simeq c$. ∎

2 LEMMA *Let K be a connected simplicial complex. Then K contains a maximal tree, and any maximal tree contains all the vertices of K.*

PROOF The collection of trees contained in K is partially ordered by inclusion. Let $\{K_j\}$ be a simply ordered set of trees in K and let $T = \cup K_j$. We show that T is a tree. Since K_j is one-dimensional, T is one-dimensional. Since $\{K_j\}$ is a simply ordered set of trees, it follows that any finite subcomplex of T is contained in some K_j. To show that T is simply connected, let $f\colon S^i \to |T|$, where $i = 0$ or 1. Then $f(S^i)$ is compact and is therefore contained in $|K_j|$ for some j. Since $|K_j|$ is simply connected, the map $f\colon S^i \to |K_j|$ can be extended to a map $f'\colon E^{i+1} \to |K_j| \subset |T|$, and $|T|$ is simply connected.

As a result, every simply ordered set of trees in K has a tree as upper bound. Zorn's lemma can be applied to yield a maximal tree in K. If T is a maximal tree of K and does not contain all the vertices of K, it follows from the connectedness of K that there is a 1-simplex $\{v_1,v_2\} \in K$ with $v_1 \in T$ and $v_2 \notin T$. Let $T_1 = T \cup \{\{v_2\}, \{v_1,v_2\}\}$. Since v_1 is a strong deformation retract of $|\{v_1,v_2\}|$, $|T|$ is a strong deformation retract of $|T_1|$. Therefore $|T_1|$ is contractible, and so T_1 is a tree strictly larger than T, contradicting the maximality of T. ∎

It follows from lemma 2 that if K is a connected complex and T is a maximal tree in K, then $K - T$ consists of simplexes of dimension ≥ 1. Because $|T|$ is contractible, any edge path in K is determined by its part in $K - T$, as we shall see below. This motivates the following definition.

Let T be a maximal tree of the connected complex K. Let G be the group generated by the edges (v,v') of K with the relations

(*a*) If (v,v') is an edge of T, then $(v,v') = 1$.
(*b*) If v, v', and v'' are vertices of a simplex of K, then $(v,v')(v',v'') = (v,v'')$.

3 THEOREM *With the notation above,* $E(K,v_0) \approx G$.

PROOF Since T is connected and contains the vertices of K, for each vertex v of K there is an edge path ζ_v in T such that orig $\zeta_v = v_0$ and end $\zeta_v = v$. If (v,v') is an edge of K, the edge path $\zeta_v(v,v')\zeta_{v'}{}^{-1}$ is a closed edge path in K at v_0. Therefore there is a homomorphism α of the free group F generated by the edges of K into $E(K,v_0)$ defined by $\alpha(v,v') = [\zeta_v(v,v')\zeta_{v'}{}^{-1}]$.

We show that α can be factored through G. If (v,v') is an edge of T, then $\zeta_v(v,v')\zeta_{v'}{}^{-1}$ is a closed edge path in T. Because T is simply connected, $[\zeta_v(v,v')\zeta_{v'}{}^{-1}] = 1$ and α sends relations of type (*a*) into 1. If v, v' and v'' are vertices of a simplex of K, then

$$\begin{aligned}[\zeta_v(v,v')\zeta_{v'}{}^{-1}] \circ [\zeta_{v'}(v',v'')\zeta_{v''}{}^{-1}] &= [\zeta_v(v,v')(v',v'')\zeta_{v''}{}^{-1}] \\ &= [\zeta_v(v,v'')\zeta_{v''}{}^{-1}]\end{aligned}$$

Therefore $\alpha((v,v')(v',v'')) = \alpha(v,v'')$, and so there is a homomorphism $\alpha'\colon G \to E(K,v_0)$ such that $\alpha'(v,v') = \alpha(v,v') = [\zeta_v(v,v')\zeta_{v'}{}^{-1}]$.

To prove that α' is an isomorphism we construct an inverse $\beta'\colon E(K,v_0) \to G$ as follows. For each closed edge path $\zeta = e_1 \cdots e_r$ let $\beta(\zeta) = e_1 \cdots e_r$, where the right-hand side is interpreted as a product in G. If ζ and ζ' are simply equivalent, then because of the relations of type (*b*), $\beta(\zeta) = \beta(\zeta')$. Therefore there is a homomorphism $\beta'\colon E(K,v_0) \to G$ such that $\beta'[\zeta] = \beta(\zeta)$.

We show that α' and β' are inverses of each other. Given an edge path $\zeta = (v_0,v_1)(v_1,v_2) \cdots (v_r,v_0)$, then $\alpha'\beta'[\zeta] = [\zeta']$, where

$$\begin{aligned}\zeta' &= \zeta_{v_0}(v_0,v_1)\zeta_{v_1}{}^{-1}\zeta_{v_1}(v_1,v_2)\zeta_{v_2}{}^{-1} \cdots \zeta_{v_r}(v_r,v_0)\zeta_{v_0}{}^{-1} \\ &\sim \zeta_{v_0}(v_0,v_1)(v_1,v_2) \cdots (v_r,v_0)\zeta_{v_0}{}^{-1}\end{aligned}$$

Since ζ_{v_0} is a closed edge path in the simply connected complex T, $\zeta_{v_0} \sim 1$ and $\zeta' \sim \zeta$. Therefore $\alpha'\beta'$ is the identity on $E(K,v_0)$.

Consider $\beta'\alpha'(v,v') = \beta(\zeta_v)(v,v')\beta(\zeta_{v'}{}^{-1})$. Since ζ_v and $\zeta_{v'}{}^{-1}$ are paths in T, they are products of edges in T. Hence $\beta(\zeta_v)$ and $\beta(\zeta_{v'}{}^{-1})$ are both equal to 1 by relations of type (*a*). Therefore $\beta'\alpha'(v,v') = (v,v')$, and since $\{(v,v')\}$ generate G, $\beta'\alpha' = 1$ on G. ■

In case K is finite, there are only a finite number of edges of K, and G is finitely generated. Similarly, there are only a finite number of relations of type (*a*) or (*b*). G is thereby represented as the quotient group of a finitely generated free group by a finitely generated subgroup. Hence we have the following corollary.

4 COROLLARY *If K is a finite connected simplicial complex, then $E(K,v_0)$ is finitely presented.* ■

5 COROLLARY *If K is a connected graph, $E(K,v_0)$ is a free group, and if T is a maximal tree in K, a set of generators of $E(K,v_0)$ is in one-to-one correspondence with the 1-simplexes of $K - T$.*

PROOF Consider the group G. Because of relations of type (a), we need only consider edges of K not in T. Every such edge e corresponds to an order of the vertices of some 1-simplex of $K - T$. The relations of type (b) imply that the oppositely ordered edge equals e^{-1} in G. Therefore the group G is generated by edges one for each 1-simplex of $K - T$. There are no relations on these generators of G, for if v, v', and v'' are vertices of a simplex of K, then at least two of them are equal. If $v = v'$ or $v' = v''$, the corresponding relation of type (b) is the trivial relation $1(v',v'') = (v',v'')$ or $(v,v')1 = (v,v')$. If $v = v''$, the corresponding relation is $(v,v')(v',v) = 1$, which, in terms of our generators, becomes $ee^{-1} = 1$. ■

6 EXAMPLE Let $J = \{j\}$ be a set and let X be the pointed space which is the sum (in the category of pointed spaces) of pointed 1-spheres $\{S_j^1\}_{j \in J}$. Each S_j^1 can be triangulated by $\dot{s}_j$, where s_j is a 2-simplex $s_j = \{v_j,v_j',v_j''\}$ and v_j corresponds to the base point of S_j^1. Then X can be triangulated by the complex K with vertices

$$\{v\} \cup \{v_j',v_j''\}_{j \in J}$$

and 1-simplexes

$$\{\{v,v_j'\}\}_{j \in J} \cup \{\{v,v_j''\}\}_{j \in J} \cup \{\{v_j',v_j''\}\}_{j \in J}$$

Let T be the maximal tree in K such that $K - T$ consists of the collection $\{\{v_j',v_j''\}\}_{j \in J}$. By corollary 5, $E(K,v)$ is a free group on generators in one-to-one correspondence with J. Therefore there is an isomorphism of $\pi(X,x_0)$, where x_0 corresponds to v, with the free group generated by J.

7 EXAMPLE Let X be the complement in $\mathbf{R}^2$ of a set of p disjoint closed disks or points. Then X has the same homotopy type as the sum of p pointed 1-spheres. Therefore the fundamental group of X is a free group with p generators.

For connected graphs the fundamental group functor is a faithful representation of the category of their underlying spaces and homotopy classes by means of groups and homomorphisms. This is summarized in the following theorem.

8 THEOREM *Let K_1 and K_2 be connected graphs and let v_0 be a vertex of K_1. Then*

(a) Any continuous map $f\colon |K_1| \to |K_2|$ is homotopic to a continuous map $f'\colon |K_1| \to |K_2|$ such that $f'(v_0)$ is a vertex of K_2.
(b) If v_0' is any vertex of K_2 and $h\colon \pi(|K_1|,v_0) \to \pi(|K_2|,v_0')$ is an arbitrary homomorphism, there exists a continuous map $f\colon (|K_1|,v_0) \to$

$(|K_2|,v'_0)$ such that $h = f_\#$.

(c) Let v'_0 and v''_0 be vertices of K_2 and assume that f_1, f_2: $|K_1| \to |K_2|$ are maps such that $f_1(v_0) = v'_0$ and $f_2(v_0) = v''_0$. Then $f_1 \simeq f_2$ if and only if there is a path ω in $|K_2|$ from v'_0 to v''_0 such that the following triangle is commutative:

$$\begin{tikzcd} & \pi(|K_1|,v_0) \arrow[dl, "f_{1\#}"'] \arrow[dr, "f_{2\#}"] & \\ \pi(|K_2|,v'_0) & & \pi(|K_2|,v''_0) \arrow[ll, "h_{[\omega]}"'] \end{tikzcd}$$

PROOF Since K_2 is connected, it is path connected, and (a) follows from the fact that the pair $(|K_1|,v_0)$ has the homotopy extension property with respect to $|K_2|$ (by corollary 3.2.5).

To prove (b), let T be a maximal tree in K_1. Let $\{s_j\}$ be the simplexes of $K_1 - T$ and for each j let $e_j = (v_j,v'_j)$ be an edge whose vertices are the vertices of s_j in some order. For each vertex v in K_1 there is an edge path ζ_v in T from v_0 to v. For each j let

$$\omega_j = |\zeta_{v_j} e_j \zeta_{v'_j}^{-1}|$$

Then $[\omega_j] \in \pi(|K_1|,v_0)$, and by corollaries 5 and 3.6.17, the set $\{\omega_j\}$ is a system of free generators of $\pi(|K_1|,v_0)$. For each j let ω'_j be a closed path in $|K_2|$ at v'_0 such that $h[\omega_j] = [\omega'_j]$. We define a continuous map

$$f\colon (|K_1|,v_0) \to (|K_2|,v'_0)$$

by $f(|T|) = v'_0$, and for each j we define $f \mid |s_j|$ by

$$f(tv'_j + (1-t)v_j) = \omega'_j(t)$$

where the points of $|s_j|$ are written in the form $tv'_j + (1-t)v_j$ for $t \in I$. f is continuous because its restriction to $|T|$ and to each $|s_j|$ is continuous. Clearly, $f_\#[\omega_j] = [\omega'_j] = h[\omega_j]$; therefore $f_\# = h$.

To prove (c), note that if $f_1 \simeq f_2$, there is a path ω in $|K_2|$ from v'_0 to v''_0 such that, by theorem 1.8.7, $f_{1\#} = h_{[\omega]}f_{2\#}$. Conversely, if $f_{1\#} = h_{[\omega]}f_{2\#}$, let T be a maximal tree in K_1 and for each vertex v of K_1 let ζ_v be an edge path in T from v_0 to v. We shall define F: $|K_1| \times I \to |K_2|$, a homotopy from f_1 to f_2, in several stages. First we set $F(x,0) = f_1(x)$ and $F(x,1) = f_2(x)$ for $x \in |K_1|$. Then F has been defined on $(|K_1| \times 0) \cup (|K_1| \times 1)$. If v is a vertex of K_1, we define $F(v,t) = ((f_1(|\zeta_v{}^{-1}|) * \omega) * f_2|\zeta_v|)(t)$ for $t \in I$. Then $F(v,0) = f_1(v)$ and $F(v,1) = f_2(v)$, and F is thus defined on $|K_1{}^0| \times I$ to be consistent with its previous definition on $(|K_1| \times 0) \cup (|K_1| \times 1)$. It only remains to extend F over $|s| \times I$ for each 1-simplex $s \in K_1$. Let v and v' be the vertices of s in some order. Then $|s| \times I$ is a square with the following product, arbitrarily associated, as boundary

$$(|(v,v')| \times 0) * (v' \times I) * (|(v',v)| \times 1) * (v \times I)^{-1}$$

F can be extended over $|s| \times I$ if and only if F maps this product into a null homotopic path of $|K_2|$. By the definition of F, the above path is sent into a path homotopic to the following product associated arbitrarily

$$\begin{aligned} f_1|(v,v')| * (f_1|\zeta_{v'}^{-1}| * \omega * f_2|\zeta_{v'}|) * f_2|(v',v)| * (f_2|\zeta_v^{-1}| * \omega^{-1} * f_1|\zeta_v|) \\ \simeq f_1|(v,v')| * f_1|\zeta_{v'}^{-1}| * (\omega * f_2(|\zeta_{v'}| * |(v',v)| * |\zeta_v^{-1}|) * \omega^{-1}) * f_1|\zeta_v| \\ \simeq f_1|(v,v')| * f_1|\zeta_{v'}^{-1}| * f_1(|\zeta_{v'}| * |(v',v)| * |\zeta_v^{-1}|) * f_1|\zeta_v| \\ \simeq \varepsilon_{f_1(v)} \end{aligned}$$

Therefore F can be extended over $|s| \times I$, and the resulting map $F\colon |K_1| \times I \to |K_2|$ will be continuous, because for each closed simplex $|s|$ of K_1 its restriction to $|s| \times I$ is continuous. Then $F\colon f_1 \simeq f_2$. ∎

It follows from theorem 8*b* that if $f\colon (|K_1|,v_0) \to (|K_2|,v_0')$ induces an isomorphism $f_\#\colon \pi(|K_1|,v_0) \approx \pi(|K_2|,v_0')$, then there is a continuous map $g\colon (|K_2|,v_0') \to (|K_1|,v_0)$ such that $g_\# = (f_\#)^{-1}$. By theorem 8*c*, it follows that $gf \simeq 1_{|K_1|}$ and $fg \simeq 1_{|K_2|}$. Hence we have the next result.

9 COROLLARY *Let K_1 and K_2 be connected graphs with v_0 a vertex of K_1 and v_0' a vertex of K_2. A continuous map $f\colon (|K_1|,v_0) \to (|K_2|,v_0')$ is a homotopy equivalence if and only if f induces an isomorphism $f_\#\colon \pi(|K_1|,v_0) \approx \pi(|K_2|,v_0')$.* ∎

The step-by-step extension procedure used to construct the homotopy F to prove theorem 8*c* is a standard method for constructing continuous maps on the space of a complex. The map is constructed on one skeleton at a time and extended over the next skeleton.

8 EXAMPLES AND APPLICATIONS

This section contains assorted results concerning the fundamental group. We begin with some applications to the theory of free groups; in particular, we show that any subgroup of a free group is free. Next we consider the effect on the fundamental group of attaching 2-cells to a space. We use the result obtained to prove that any group is isomorphic to the fundamental group of some space. Finally, we describe how the fundamental group of a surface can be represented by means of generators and relations.

If K is a simplicial complex and $\alpha \in |K|$ has carrier s (that is, $\alpha \in \langle s \rangle$), then for any subdivision K' of $\dot{s}$ the simplicial complex $K' * \alpha$ is a subdivision of $\bar{s}$ (by lemma 3.3.8). It follows that a modified barycentric subdivision of K can be constructed whose vertices are α and the barycenters of simplexes of K other than s. Therefore there is a subdivision of K having α as a vertex, and we have the following result.

1 LEMMA *If $\alpha \in |K|$, there is a subdivision K' of K having α as a vertex.* ■

2 THEOREM *A polyhedron is locally contractible.*

PROOF In view of lemma 1, it suffices to prove that if v is a vertex of a simplicial complex K, every neighborhood U of v in $|K|$ contains a neighborhood V of v deformable in U to v. Let U be a neighborhood of v and let $A = \text{st } v$. Define $F: A \times I \to |K|$ by

$$F(\alpha,t) = tv + (1 - t)\alpha$$

Then F is a deformation of A in $|K|$ to the point v, and $F(v \times I) = v \in U$. Therefore there is some neighborhood V of v in A such that $F(V \times I) \subset U$. Because $A = \text{st } v$ is open in $|K|$, V is a neighborhood of v in $|K|$. Since $F|\, V \times I$ is a deformation of V in U to v, $|K|$ is locally contractible. ■

It follows from theorem 2 that the theory of covering projections applies to polyhedra, and corresponding to any subgroup of the fundamental group of a polyhedron there is a covering projection. We show that any covering projection of a polyhedron corresponds to a simplicial map.

3 THEOREM *Let $p: \tilde{X} \to X$ be a covering projection, where X is a polyhedron. Then $\tilde{X}$ is a polyhedron, of the same dimension as X, in such a way that p corresponds to a simplicial map.*

PROOF Assume that $p: \tilde{X} \to |K|$ is a covering projection. For any simplex $s \in K$ the closed simplex $|s|$ is simply connected. It follows from the lifting theorem that the inclusion map $|s| \subset |K|$ can be lifted to a map $|s| \to \tilde{X}$, and it follows from the unique lifting theorem that two such liftings are either identical or have disjoint images. Hence there are as many liftings of $|s|$ as sheets of $\tilde{X}$ over $|s|$.

Define a simplicial complex $\tilde{K}$ to have the collection $\{p^{-1}(v) \mid v \text{ is a vertex of } K\}$ as vertex set and to have simplexes $\{\tilde{s}\}$, where $\tilde{s} = \{\tilde{v}_0, \ldots ,\tilde{v}_q\}$ is a simplex of $\tilde{K}$ if and only if there is a simplex $s = \{v_0, \ldots ,v_q\}$ in K and a lifting $f_{\tilde{s}}: |s| \to \tilde{X}$ of $|s|$ such that $f_{\tilde{s}}(v_i) = \tilde{v}_i$ for $0 \le i \le q$ [in which case $s = p(\tilde{s})$ and $f_{\tilde{s}}$ are both unique]. Then $\tilde{K}$ is a simplicial complex and has the same dimension as K. If $\tilde{s}_1$ is a face of $\tilde{s}$, then $p(\tilde{s}_1)$ is a face of $p(\tilde{s})$ and $f_{\tilde{s}} \mid |p(\tilde{s}_1)| = f_{\tilde{s}_1}$. Therefore the collection $\{ f_{\tilde{s}}\}_{\tilde{s} \in \tilde{K}}$ defines a continuous map $f: |\tilde{K}| \to \tilde{X}$ such that

$$f(\Sigma\alpha_i\tilde{v}_i) = f_{\tilde{s}}(\Sigma\alpha_i p(\tilde{v}_i)) \qquad \Sigma\alpha_i\tilde{v}_i \in |\tilde{s}|$$

Let $\varphi: \tilde{K} \to K$ be the simplicial map $\varphi(\tilde{v}) = p(\tilde{v})$. Then there is a commutative triangle

$$\begin{array}{ccc} |\tilde{K}| & \xrightarrow{f} & \tilde{X} \\ & {\scriptstyle |\varphi|}\searrow \quad \swarrow{\scriptstyle p} & \\ & |K| & \end{array}$$